GEOMETRICAL DRAWING

SECOND PART

Geometrical Drawing

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FOR THE USE OF CANDIDATES FOR ARMY EXAMINATIONS

BY

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PREFACE.

THIS book is intended for those who have already acquired a knowledge of the principles of Geometrical Drawing. It is, in the main, an extension of the author's earlier work on the subject, and the same lines are followed throughout. The titles and divisions of the earlier chapters are identical with those of the First Part, and the problems given are, in many cases, extensions of the more elementary constructions. As in the First Part, proofs of the problems are indicated, answers are given, and the figures, to which great attention has been paid, are drawn exactly in the form in which they should be reproduced.

As far as possible, the problems have been divided into systematic groups, so that each group depends on the same general principles of construction. Isolated constructions, which have no application to other problems, have, in general, been omitted. On the other hand, separate constructions are given for the regular figures from nine to twelve sides. A large number of Geometrical Patterns has been printed. Some of these have been set in Army Examinations, but the majority

are original, and have been carefully selected to illustrate important principles—in particular, the change of curve produced by tangency.

A chapter has been added dealing fully with the approximate construction of Conic Sections and other curves.

I desire to express my thanks to R. A. H. MacFarland, Esq., of Repton School, and H. Richardson, Esq., and P. A. Thomas, Esq., of Sedbergh School, for many valuable suggestions; also to many of my pupils, and, in particular, to three who produced original constructions which I have adopted, H. W. Mackworth, O. W. Brinton, and H. Farrant.

W. N. WILSON.

RUGBY, *August*, 1896.

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INTRODUCTION.

In the following chapters a knowledge of the Elements of Geometrical Drawing is assumed. The figures in the text should be carefully reproduced exactly in the form in which they are given, but on a larger scale. The geometrical proofs indicated should be carefully studied at the same time as the figures are copied. Reference is constantly made to problems in the author's earlier work on Geometrical Drawing under the title "First Part". Each of these should be revised, where it is mentioned.

In drawing a figure, it must be borne in mind that, since a geometrical proof is seldom required, the construction of any problem must be made clear in the figure itself. For this purpose, it is best to represent original data by fine continuous lines, lines of construction by dotted lines, and final results by darker continuous lines. This is more readily done if the figure is to be inked in, than if it

is only to be drawn in pencil; for, sometimes, it is difficult to draw a dotted pencil line, when some point or points have, afterwards, to be found on it. This is especially the case with arcs of circles; indeed, dotted arcs, in pencil, should only be used to denote that lines, radiating from the same point, are equal. Of course, if the figure is to be inked in, it will be possible to draw dotted ink lines for all the lines of construction, including circles, and so increase the clearness of the figure. Good dotted lines are of the greatest importance; they should never be hurried; indeed, every line should be drawn slowly. The use of india-rubber should be avoided, except in the case of the intersecting arcs used to indicate bisection, and, then, only to make the two arcs into a cross with arms of equal length. India-rubber, too, should be unnecessary, when ink is used, for a figure should be drawn, in pencil, with the same care, whether it is to be inked in or not. It is a common complaint that a pen will not draw. The remedy is to clear out the ink and fill it again; but it is important to see that all is right before beginning to ink in, for, if the lines drawn are to be the

same thickness throughout, the screw of the pen must not be touched.

Keep all your instruments scrupulously clean. It is a good plan to keep a piece of chamois-leather in your box, and rub off finger-marks before you put the instruments away. Make it a rule always to sharpen your pencils before you begin to work. A pencil is not sharp if you can see the point. When vulcanite set squares are dirty, they should be cleaned with bread: if you rub them, you charge them with electricity, and so help to collect the dust.

CHAPTER I.

PRELIMINARY CONSTRUCTIONS.

ERRATA.

Page 14, Ex. 11, *after* middle points *read* of the sides.

Page 14, Ex. 17, *for* $a - b$ *read* $b - a$.

Page 105, line 4, *for* longer *read* shorter.

Page 106, line 5, *for* Measure . . . division *read* Find the area of one of the parts.

Page 136, line 4, *for* $2\frac{1}{2}''$ *read* $2{\cdot}60''$.

Page 147, line 7, *for* circle *read* ellipse.

Answers.

Page 183, Chap. III., 12, *for* $1{\cdot}24''$ *read* $1{\cdot}62''$.

Page 183, Chap IV., 12, *for* $1{\cdot}25''$ *read* $\cdot 63''$.

Page 184, Chap. VI., 7, *for* $\frac{1}{44000}$ *read* $\frac{1}{1222}$.

Page 184, Chap. VI., 21, *for* 6·62 *read* 6·01.

Page 185, Chap. X., 10, *for* $\cdot 84''$ *read* ·39 sq. ins.

Page 187, Paper IX., 3, *for* 11 *read* 1·1.

Page 188, Paper XI., 6, *delete* Reduce . . . (Euc. VI., 1) *and put as follows:* Place the two triangles so that their bases form a continuous straight line. Reduce to an equivalent triangle. (Euc. I., 37, and Prob. 1, Chap. X.) Hence, by the same problem, altitude of required triangle $= 2{\cdot}54''$.

Note.—The construction is impossible if AB is perpendicular to CD.

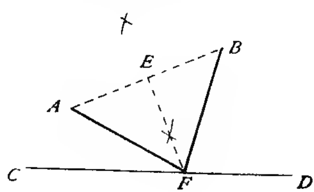
CHAPTER I.

PRELIMINARY CONSTRUCTIONS.

THE following are general constructions which will sometimes be found useful.

Problems.

1. *To find a point in a given straight line equidistant from two given points.*



A, B are the given points, CD the given straight line.

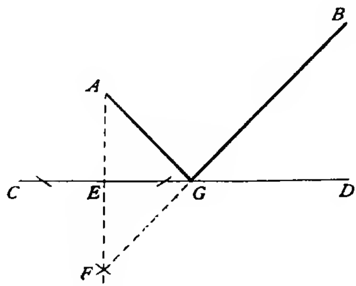
Join AB and bisect it at right angles by EF, meeting CD at F. Join AF, BF.

Then $AF = BF$.

PROOF. Euclid I. 4 applied to the triangles AEF, BEF.

Note.—The construction is impossible if AB is perpendicular to CD.

2. *To find a point in a given straight line so that the lines joining it to two given points make equal angles with the given straight lines.*



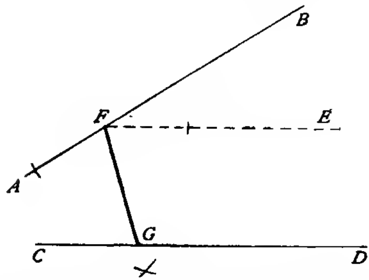
With A, one of the given points A, B, for centre and with any radius describe an arc cutting CD in two points; with these two points for centres and *the same radius* draw arcs cutting at F. Join FB, cutting CD at G. Join AG.

$$\text{Then } \llcorner AGC = \llcorner BGD.$$

Proof. By Euclid I. 8 and I. 4 AF is perpendicular to CG and is bisected at E (see figure).

$$\text{Then } \llcorner AGE = \llcorner FGE \qquad \text{(Euc. I. 4.)}$$
$$= \llcorner BGD \qquad \text{(Euc. I. 15.)}$$

3. *To draw a straight line making equal angles with two given straight lines, whose point of intersection is unknown.*



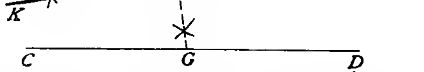
Draw EF parallel to CD, one of the given lines, meeting AB, the other line, at F. Bisect the angle AFE by FG.

Then FG makes equal angles with AB, CD.

PROOF. $\angle CGF = \angle GFE$ (Euc. I. 29.)

$= \angle AFG$ (Construction.)

4. *To draw a straight line which will bisect the angle between two given straight lines, whose point of intersection is unknown, and which will pass through that point of intersection.*



Draw FG, by the previous construction, making equal angles with AB, CD, the given straight lines.

Draw HK, bisecting FG at right angles.

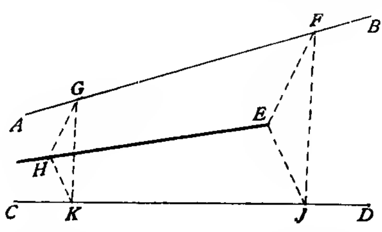
Then HK will be equally inclined to AB and CD, and will pass through their point of intersection.

Proof. If O is the point of intersection of AB, CD; then, if OH be joined,

$$OF = OG \qquad \text{(Euc. I. 6.)}$$
$$\llcorner FOH = \llcorner GOH$$
$$\text{and } \llcorner OHF = \llcorner OHG \qquad \text{(Euc. I. 8.)}$$
$$= \text{right angle.}$$

That is HK coincides with OH, the bisector of the angle BOD.

5. *Through a given point to draw the straight line which will pass through the point of intersection, supposed unknown, of two given straight lines.*

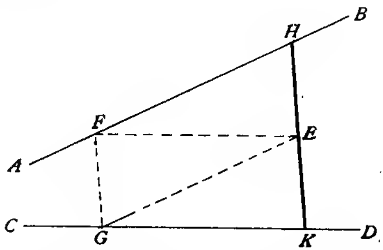


Take any two points F, G, as far apart as possible, on AB, one of the given straight lines AB, CD. Then if E is the given point, join EF and draw GH parallel to it. Draw any straight line FJ, and draw GK parallel to it, as in the figure. Join EJ, and draw KH parallel to it, cutting GH in H. Then the line joining EH will pass through the intersection of AB, CD.

PROOF. The point of intersection of AB, CD, is a "centre of similitude" of the triangles EFJ, HGK. (*See* page 29.)

$\therefore$ EH passes through this point.

6. *To draw, through a given point, a straight line to be terminated by two given lines and to be bisected at the given point.*



Draw EG, EF parallel to AB, CD, the given lines, through E, the given point. Let these meet AB, CD in F, G respectively. Draw HEK parallel to FG.

Then HK is bisected at E.

PROOF. Both $EKGF$ and $HEGF$ are parallelograms.

$$\therefore EK = GF$$
$$= HE$$

COR. This construction can be used for solving the previous problem, viz., *To draw a straight line through E to the point of intersection of AB, CD.*

Draw HK as above, bisected at E. Draw another line parallel to HK, and bisect it. Join the points of bisection of the two lines. (*See also* Chap. v., Prob. 3.)

EXERCISES—I.

The following simple problems are added for the student's own solution. The first three constructions involve the bisecting of one or more angles, the next three depend upon the properties of parallel straight lines.

1. Draw a straight line parallel to the base of an isosceles triangle so that the length intercepted on it by the two sides is equal to each of the parts of the sides cut off between the line and the base.

2. Draw a straight line parallel to the base of a triangle so that the part intercepted by the sides is equal to the sum of the intercepts on the sides between the line and the base of the triangle.

3. Repeat the previous problem, but make the constructed line equal to the difference of the two intercepts on the sides.

4. Through a given point draw a straight line so that the part intercepted between two given parallel lines is a given length.

5. Draw a straight line equally inclined to two given straight lines so that the part intercepted between them is of a given length.

6. Draw a straight line through a given point so that the part intercepted between two given lines is equal to the part between the given point and the nearer of the two given lines.

CHAPTER II.

TRIANGLES.

Though a triangle can, in general, be constructed if any three properties connected with it are given, it is seldom that triangles can be grouped together so that the same method of construction may be applied to all problems in the same group. The most important of such groups is that which involves the construction, in each case, of an "Auxiliary Triangle," as given below. In some groups only the given properties can be classified while the solutions are more or less independent, such as those involving the "Sum or Difference of Sides". (*See* p. 11.) The majority of cases, however, require each a separate construction. Some of these depend on simple geometrical properties, like the triangles given in Chapter II. (First Part). Others, involving more complicated geometrical reasoning, require isolated constructions, and so are less suitable for geometrical drawing.

Of such isolated constructions the following are a few of the more simple cases :—

1. *To construct a triangle, ABC, having given the middle points of its sides, D, E, F.*

The line joining any two of the points *D*, *E*, *F* is parallel to the side of *ABC* which passes through the third.

2. *To construct a triangle, ABC, having given the feet of the perpendiculars D, E, F from the angular points to the opposite sides.*

If *D* is the point on *BC*, then *BC* bisects the exterior angle between *ED*, *DF*, so that *BC* can be drawn, and similarly the other two sides.

3. *To construct a triangle, ABC, given the middle points D, E of the sides BC, CA, and F the foot of the perpendicular from C to AB.*

Draw the circle circumscribing the triangle *DEF*. Draw *FG* parallel to *DE*, meeting this circle at *G*. Then, by the properties of the "Nine-points Circle," *G* is the middle point of the side *AB*. Now proceed as in Problem 1.

4. *Given the vertical angle, A, of the triangle ABC, one of the sides containing it, AB, and the perpendicular AD from A to the base BC, construct the triangle.*

Construct the angle *BAC*, and mark off the given length *AB*. On *AB* describe a semicircle. With *A* for centre and the perpendicular *AD* for radius draw an arc cutting the semicircle at *D*. Join *BD*, cutting *AC* at *C*.

Auxiliary Triangles.

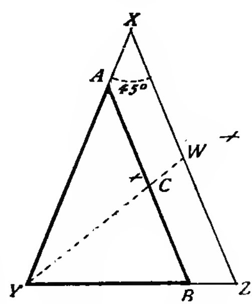
In some cases, by using *two* of the given conditions only, it is possible to draw a triangle similar to the one to be constructed. Such a triangle is called an "Auxiliary Triangle". When this is constructed, the properties of Similar Triangles will introduce the third condition.

Cases of Auxiliary Triangles have already occurred in Chapter II. (First Part), Problems 5, 9, 10, 11, 17, 18.

The following problems illustrate the general method to be followed:—

5. *Draw an isosceles triangle with a vertical angle of 45°, and the straight line joining one end of the base to the middle point of the opposite side 2·7″ long.*

Draw the angle YXZ equal to 45°. Mark off *any* equal lengths, XY, XZ. Join YZ.

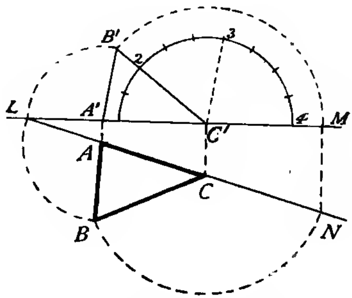


Then XYZ is an Auxiliary Triangle, for it is similar to the triangle required.

The third condition involves the straight line joining an extremity of the base to the middle point of the opposite side. Draw this line YW in the triangle XYZ. On YW mark off YC equal to the required length, 2·7″, and through C draw AB parallel to XZ, meeting YX, YZ in A and B.

Then AYB is the required triangle, for it is similar to XYZ and satisfies the conditions given.

6. *To draw a triangle, given the perimeter, and the angles in a given proportion (say 2 : 3 : 4).* (*See* p. 18, First Part.)



Draw a semicircle with centre C', and, *by trial,* divide the circumference into $2 + 3 + 4$ equal parts. (*See* Note, p. 24.)

Join $C'2$, and at B', any convenient point on it, draw $B'A'$ parallel to $C'3$. $A'B'C'$ is an auxiliary triangle, having its angles in the given proportion. Mark off LM, the perimeter of this triangle along $A'C'$, as shown in the figure. Through L draw LN equal to the given perimeter, and construct the triangle ABC, as shown in the figure. (*See* p. 18, First Part.)

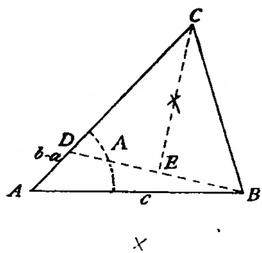
ABC is the required triangle.

Note.—The above problem shows how the whole construction can be explained in the figure itself. It may be considered as a type of how, in geometrical drawing, figures should be made self-explanatory.

Triangles in which the Sum or Difference of the Sides is Given.

7. *To construct a triangle, given the base, a base angle, and the difference of the sides (c, A, $b - a$).*

Construct BAD equal to the angle A. Mark off $AB = c$ and $AD = b - a$. Join DB. Bisect BD at right angles by EC. Join BC.

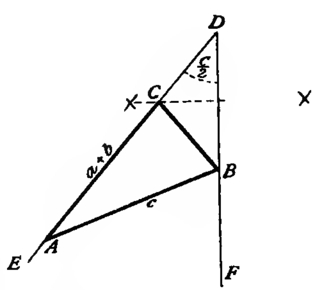


Then ABC is the triangle.

PROOF. Both construction and proof are the same as when $a + b$ is given instead of $b - a$. (*See* p. 15, First Part.)

8. *To construct a triangle, given the base, the vertical angle, and the sum of the sides (c, C, $a + b$).*

Construct EDF equal to *half* the vertical angle. On DE mark off DA equal to $a + b$. With A for centre and c for radius describe an arc, cutting DF in B. Join AB. Bisect BD by a perpendicular cutting AD at C. Join CB.



Then ABC is the triangle required.

PROOF. $CB = CD$;

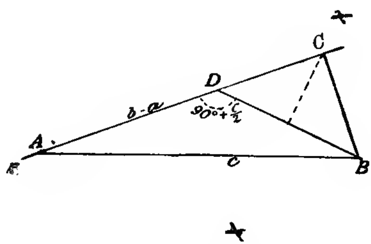
$\therefore \quad AC + CB = AD$

$= a + b$

also $\angle ACB = \angle CDB + \angle CBD$

$= 2 \angle CDB =$ given angle.

9. *To construct a triangle, given the base, the vertical angle, and the difference of the sides (c, C, b - a).*



Construct EDB equal to $90° + \frac{C}{2}$, and make the rest of the construction as in the previous problem.

PROOF. $\angle ACB = 180° - 2 (\angle CDB)$

$= 180° - 2 (180° - ADB)$

$= 180° - 2 (90° - \frac{C}{2})$

$= C.$

EXERCISES—II.

Exercises on Auxiliary Triangles.

1. Draw the triangle whose base angles are 60° and 45°, and whose altitude is $1\frac{1}{2}''$. Measure the base. (For alternative method, *see* p. 20, First Part.)

2. Draw the triangle whose altitude and base are equal and whose vertical angle is 30°, the length of the longer side being 2·5″. Measure the base. (*See* p. 16, First Part.)

3. Draw the triangle in which the sides are proportional to 2 : 3 : 4, the perpendicular on to the longest side from the opposite vertex being 1·5″. Measure the shortest side.

4. Draw the triangle in which the base is a third of the sum of the two sides, and one of the base angles is 60°, while the side opposite the given angle is 2″. Measure the other side.

5. Draw the triangle in which the base angles are 75° and 45° and the sum of the two sides is 4″. Measure the longer side.

6. The sides of a triangle are proportional to 3 : 4 : 5, and the radius of the escribed circle touching the shortest side is 1″. Draw the triangle and give the length of the shortest side.

7. Draw the right-angled isosceles triangle in which the perimeter is 7″, and give the length of one of the equal sides.

8. Draw the right-angled isosceles triangle in which the sum of the hypotenuse and one side is 5″. Measure the hypotenuse.

9. One side of a triangle and the base are in the proportion of 3 : 4, and the angle included between them is 30°. The altitude is $1\frac{1}{2}$″. Draw the triangle and measure the longer of the two sides.

10. Draw the right-angled triangle in which the hypotenuse is double of the difference of the sides and the shorter side is $1\frac{1}{2}''$. Measure the hypotenuse.

Miscellaneous Examples on the Construction of Triangles.

(*Note.*—In each case measure the shortest constructed side.)

11. Draw the triangle ABC, given that $A = 75°$, $B = 45°$, $a = 1{\cdot}5''$, and draw the triangle of which A, B, C are the middle points.

12. Construct the right-angled triangle ABC in which $C = 90°$, $BC = 2{\cdot}25''$, and the perimeter $= 5{\cdot}25''$.

13. Construct the right-angled triangle ABC in which $C = 90°$, $AB = 2{\cdot}25''$, and the perimeter $= 5{\cdot}25''$.

14. Construct the triangle in which the base $= 3''$, a base angle $= 30°$, and the difference of the sides $= 1{\cdot}5''$.

15. Construct the triangle in which the base is $4''$, one of the sides $3''$, and the vertical angle $120°$.

16. Construct the triangle ABC, given $c = 2''$, $C = 60°$, and $a + b + c = 5{\cdot}5''$.

17. Construct the triangle ABC, given $c = 2{\cdot}5''$, $C = 45°$, $a - b = 1''$.

18. The sides of a triangle are proportional to $3 : 4 : 5$ and the perimeter $= 6''$. Construct the triangle.

19. The angles of a triangle are proportional to $3 : 4 : 5$ and the perimeter $= 6''$. Construct the triangle.

20. Construct a triangle having each of the base angles double of the vertical angle, and the base $1{\cdot}75''$ long.

CHAPTER III.

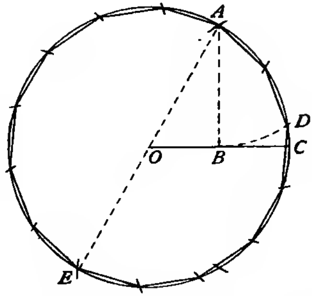
REGULAR FIGURES.

DEFINITION. A Nonagon has 9; a Decagon, 10; a Hendecagon, 11; and a Dodecagon, 12 sides.

Special constructions for the Regular Figures, which have from three to eight sides, will be found in Chapter III. (First Part). The constructions for figures of nine, ten, eleven, and twelve sides are given below. If a regular figure of a larger number of sides is required to be inscribed in a given circle, it must be constructed either by dividing the circumference, by trial, into the required number of equal parts, or by using the construction for one of the regular figures given here, or in the first part, and then sub-dividing the equal arcs so found.

If the number of sides is *the double of an odd number*, the points required on the circumference of the circle can be readily found by using the construction for determining the side of a figure of *half the number of sides*, and then stepping out that side *from opposite ends of the same diameter*. Thus:

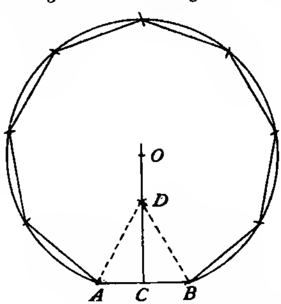
1. *To inscribe a Regular Figure in a circle when the number of sides is the double of an odd number (e.g., 14).*



Construct AD, the side of the regular heptagon inscribed in the circle. (*See* p. 31, First Part.)

Draw the diameter AE, and step out AD round the circumference both from A and E, and join.

2. *To draw a Regular Nonagon on a given line.*

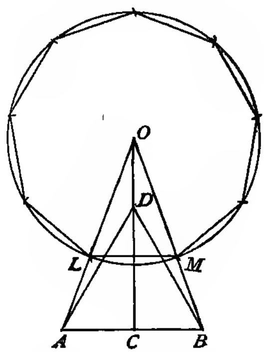


On the given line, AB, draw the equilateral triangle ADB and the perpendicular DC. On CD produced step out DO equal to AC.

Then O is the centre of the circle which will circumscribe the nonagon.

Proof. This depends on Trigonometry and is very approximate. The angle AOC is $20^\circ\ 6'$, instead of the true value 20°, and is true for practical purposes.

3. *To inscribe a Regular Nonagon in a circle.*



Draw any equilateral triangle, ADB, and the perpendicular DC. On CD produced step out DO equal to AC.

Join AO, BO, and with O for centre, and radius that of the given circle, draw a circle cutting AO, BO at L, M.

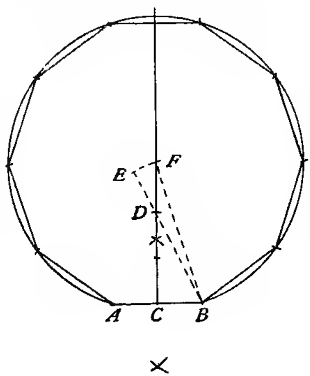
Then LM is the side of the regular nonagon inscribed in the given circle.

Proof. *See* previous problem.

Note.—In stepping out the side of a regular figure on the circumference of a circle it is best to step out half the sides in one direction and half in the opposite direction from the

starting point. Thus any error in the construction of the side is better distributed round the circumference. (*See also* Note on page 24.)

4. *To draw a Regular Decagon on a given line.*



The vertex of the regular pentagon, of which the given line AB is the base, is the centre of the circle circumscribing the decagon. (*See* Chap. III., Prob. 5 (a), First Part.)

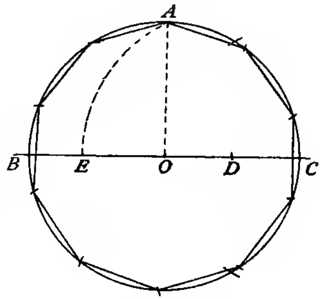
Thus, on CD, bisecting AB at right angles, mark off AC twice; and on BD produced mark off AC.

With centre B and radius BE draw an arc cutting CD at F.

Then F is the centre, and BF the radius, of the circle which circumscribes the required decagon.

Proof. This depends on Trigonometry, and is theoretically true.

5. *To inscribe a Regular Decagon in a circle.*



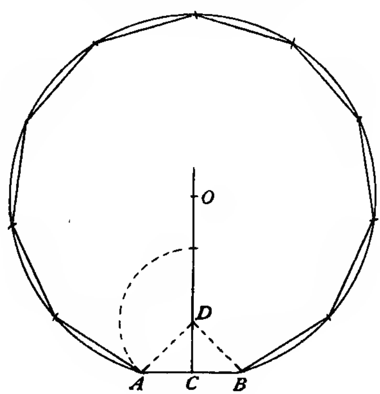
Proceed as in the construction for inscribing a regular pentagon. Thus through the centre O draw the radius OA perpendicular to the diameter BC. Bisect OC at D, and with D for centre and radius DA draw an arc cutting OC at E.

Then OE is the side of the inscribed decagon, which may be stepped out round the circumference.

PROOF. This depends upon Trigonometry.

Alternative Construction.—The decagon can also be constructed as on page 16 by stepping out AE, the side of the pentagon, from opposite ends of the diameter through A.

6. *To draw a Regular Hendecagon on a given line.*

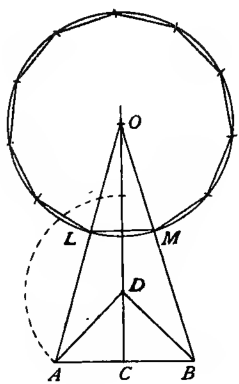


On the given line, AB, draw the right-angled isosceles triangle ADB and the perpendicular DC. On CD produced step out DO equal to $AD + DC$.

Then O is the centre of the circumscribing circle.

PROOF. This is very approximate. The angle AOC is $16° 19' 30''$, instead of the true value $16° 21' 50''$, and is, therefore, practically true.

7. *To inscribe a Regular Hendecagon in a circle.*



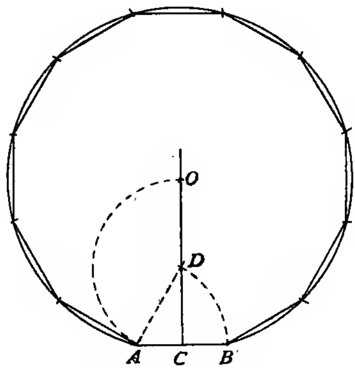
Draw any right-angled isosceles triangle ADB and the perpendicular DC. On CD produced step out DO equal to $AD + DC$.

Join AO, BO, and with O for centre and radius of given circle for radius draw a circle cutting AO, BO, at L, M.

Then LM is the side of a regular hendecagon inscribed in the given circle.

PROOF. *See* previous problem.

8. *To draw a Regular Dodecagon on a given line.*



Draw CD the perpendicular of the equilateral triangle which can be drawn on AB, and produce CD to O so that $OD = DA$.

Then O is the centre of the circle which will circumscribe the dodecagon.

Proof. If AD, AO be joined, then

$$\angle ADC = 30^\circ$$

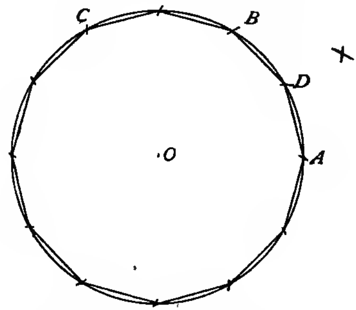
$$\text{also} \quad \angle AOD = \angle OAD$$

$$= \tfrac{1}{2} \angle ADC = 15^\circ$$

$$\therefore \quad \angle AOB = 30^\circ = \tfrac{1}{12} \times 360^\circ.$$

i.e., $\angle AOB$ is the angle subtended by the side of a regular dodecagon inscribed in a circle of which O is the centre.

9. *To inscribe a Regular Dodecagon in a circle.*



Step out the radius OA round the circumference so as to find the points A, B, C, etc., of a regular hexagon.

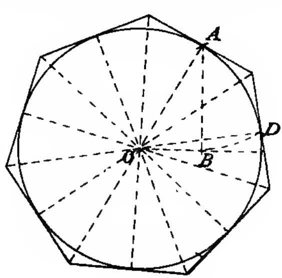
Bisect the arc AB at D, and step out AD, checking, for accuracy, at the six points already found.

Join AD, DB, etc.

Alternative Construction.—Step out the radius round the circumference from the extremities of two diameters at right angles to one another.

10. *To describe a Regular Figure about a circle.*

The figure given represents the construction for a regular heptagon described about a circle.



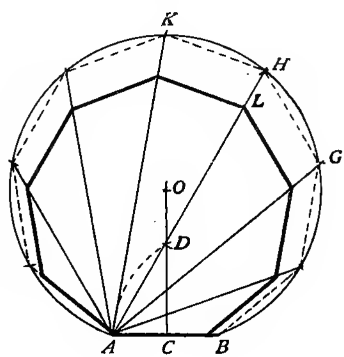
First find the points on the circumference belonging to the regular figure of the same number of sides *inscribed* in the circle. Join these points to the centre, and draw perpendiculars to the radii thus formed through the points on the circle.

If the figure to be described has an odd number of sides, the radii drawn should be produced through the centre and beyond the circle, for a point of the figure will lie on each of these produced lines.

The Use of an Auxiliary Regular Figure.

11. *To draw a Regular Figure when a diagonal is given.*

Construct a regular figure of the given number of sides (*e.g.*, nine), and of any convenient size, either in a circle or on a line.



On the diagonal AH mark off AL equal to the length given for that diagonal. Through L draw parallels to HG, HK, to meet the diagonals AG, AK. Through the points so found draw parallels again to the next sides, and so on till the figure is completed as shown.

Proof. Euc. vi. 18. *See also* "Centres of Similitude" below.

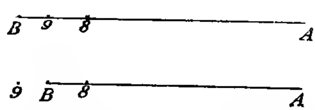
Note 1.—The construction is identical if the length of any other diagonal, such as AG, is given.

Note 2.—By a similar method a regular figure can be drawn if other lengths are given, *e.g.*, *given the sum of a side and a diagonal.* In this case the given length is divided proportionally to the side and diagonal of any regular figure of the given number of sides, drawn as above, and the figure can be completed in the same way.

It will be noticed that the principles are the same as those used in the case of Auxiliary Triangles (p. 9).

NOTE.—ON DIVISION BY TRIAL AND STEPPING OUT.

Suppose that a straight line AB, or a circle, is to be divided into a given number of equal parts, say 9; and suppose that 8, 9 represent the positions of the points of the dividers for the last division, when this is so nearly true that $B9$ is less than the last true division.



Then $B9$ is the error repeated *nine* times. Therefore keeping one point of the dividers fixed at 8, move the other point through *a ninth* of the distance $B9$. This can be done by estimation, if the point 9 is marked on the paper, and it will be found that the number of divisions can now be stepped out very accurately.

When a measured length is stepped out on the circumference of a circle, test the accuracy first by stepping out, *without marking the paper.* If the error is comparatively large, correct as indicated above, and try again. If the error is now very small, complete accuracy may be obtained by stepping out *just on the outside or inside of the circumference*, according as the error is too large or too small.

So, too, a small error in the divisions stepped out on a straight line may be corrected by putting the points of the dividers alternately on opposite sides of the line.

EXERCISES—III.

(*Note.*—In each case measure and write down the length of the side, or the radius of the circumscribing circle, of the figure drawn.)

In circles of $1\frac{1}{2}''$ radius inscribe the following regular figures :—

1. A nonagon. 2. A decagon.
3. A hendecagon. 4. A dodecagon.

On lines of length 1″ draw the following regular figures :—

5. A nonagon. 6. A decagon.
7. A hendecagon. 8. A dodecagon.

About circles of 1″ radius circumscribe the following regular figures :—

9. A hexagon. 10. A heptagon.

11. Draw the regular heptagon whose shorter diagonal is 1·5″.

12. Draw a regular octagon, given that the diagonals joining its alternate angular points are each 3″ long.

13. Draw a square, given that its diagonal and one side are together equal to 6″.

In a circle of 2″ radius inscribe regular figures of the following number of sides :—

14. Eighteen sides. 15. Fifteen sides.

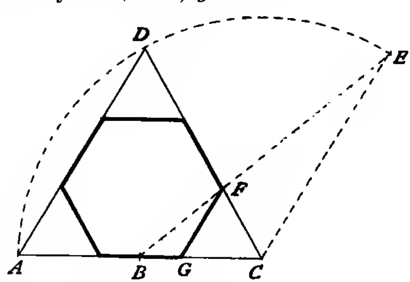
CHAPTER IV.

FIGURES INSCRIBED IN SYMMETRICAL FIGURES.

THE construction given in Chapter IV., First Part, for inscribing one regular figure in another requires that the number of sides of the inscribed figure should be not greater than the number of sides of the figure in which it is inscribed. By a slight modification it may be used to include the following problem.

1. *To inscribe in a Regular Figure a Regular Figure of double the number of sides having all its angular points on sides of the first figure.*

For example, to "*Inscribe a Regular Hexagon in an Equilateral Triangle*".



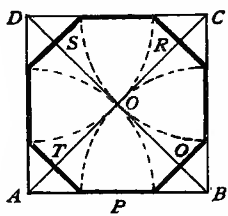
We can consider the equilateral triangle ACD as being a symmetrical figure of six sides, if we take the *middle points of the sides* (*e.g.*, B) as being points of the figure. Then AB, BC are consecutive sides, while AC is the diagonal cutting off the vertex B.

The construction is now the same as in Problem 2, p. 35, First Part, B being vertex and AC diagonal.

Make the angle ACE equal to the angle of a regular hexagon, that is 120°, and make CE equal to CA. Join BE, cutting CD in F. Draw FG parallel to CE.

Then FG is a side of the inscribed regular hexagon, which can be stepped out.

Similarly, any other regular figure can be inscribed in another of half the number of sides. The case, however, of the regular *Octagon in a Square* can be more simply dealt with in the following way :—

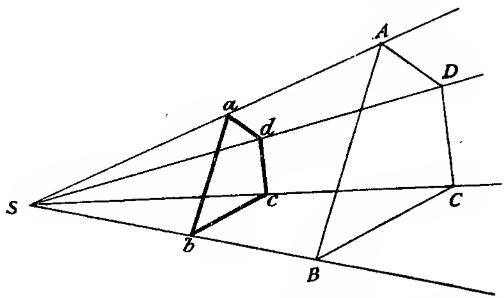


Draw the diagonals AC, BD intersecting in O. With AO for radius, and A, B, C, D for centres, draw arcs intersecting the sides. If the points so found are joined the figure formed is a regular octagon.

The more general problem, "*To inscribe in a Regular Figure of a certain number of sides another Regular Figure with a greater number of sides*," subject to the condition that as many points as possible of the latter shall lie upon sides of the former, depends upon the use of "Centres of Similitude," of which the construction used above for the hexagon is a particular case.

CENTRES OF SIMILITUDE.

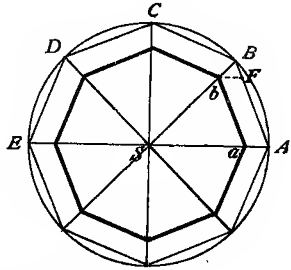
It can be proved that if two similar figures have their corresponding sides parallel, then the straight lines joining all the pairs of corresponding points will meet in one and the same point, which is called a centre of similitude.



Thus, in the figure, S is a centre of similitude of the similar figures $ABCD$, $abcd$.

Hence we have a construction for drawing one figure similar to another; for instance:—

2. *To draw a Regular Octagon, whose side is a given length, concentric with a given Regular Octagon.*



Here S, the centre of the given octagon $ABCDE$ etc., must be taken for centre of similitude, and the angular points of the required octagon will lie on the lines SA, SB, etc.

On AB, or AB produced, mark off AF equal to the side of the required octagon. Draw Fb parallel to SA, and ba parallel to BA.

Then ab is a side of the required octagon, which can be stepped out.

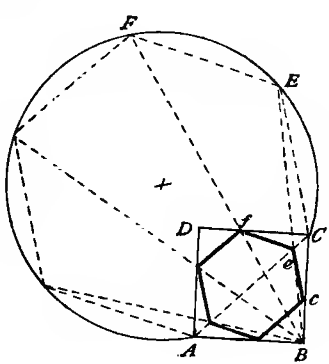
Note.—For instances of the use of centres of similitude, *see* First Part, pages 24, 25, 92, and 109.

FIGURES OF A GIVEN SHAPE INSCRIBED IN OTHERS OF A SMALLER NUMBER OF SIDES.

The principles of Centres of Similitude can be used for inscribing one figure in another. The three following examples give the general method for all. It is to be used when the number of sides of the inscribed figure is *larger* than the number of sides of the figure in which it is to be inscribed. If the number of sides of the inscribed figure is smaller, or double, the constructions given in Chapter IV., First Part, and on page 27 will be found more convenient.

(1) Inscribed Figures when an *angular point* of the given figure is a centre of similitude.

3. *To inscribe a Regular Hexagon in a Square, so that angular points of the Hexagon may fall upon sides of the Square.*



If we decide that two of the points of the hexagon shall fall on the sides *AB*, *BC*, then *B* must be chosen for centre of similitude.

Now, by symmetry, the side joining those two points must be parallel to *AC*.

If, then, we draw a regular hexagon on *AC*, and join its angular points *E*, *F*, etc., to *B*, it follows that the angular points of the inscribed hexagon must lie on *BC*, *BE*, *BF*, etc.

Therefore, from *f*, where *FB* cuts the square, draw *fe* parallel to *FE* to meet *BE* in *e*. Then *fe* is a side of the inscribed hexagon, which can be completed by stepping out *fe* as in the figure, or by drawing *ec* parallel to *EC*, and so on.

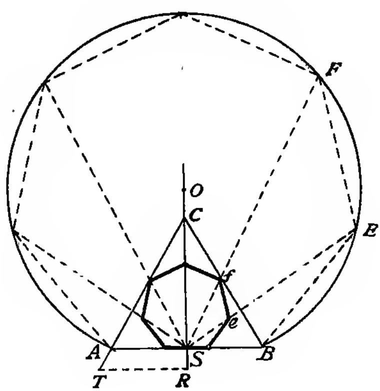
Note.—It will be noticed that the point where *FB* cuts the square is taken for the first point of the inscribed hexagon, and not the point where *EB* cuts it. If the latter had been taken, then the side drawn parallel to *EF* would have fallen *outside* the square, and, though we should still have got a regular hexagon, it would not have been inscribed in the square. Care must be taken to select for first point of the inscribed figure that one which gives none of the points of the figure, which is to be drawn, outside the given figure.

(2) Inscribed Figures when the *middle point of a side* of the given figure is a centre of similitude.

The previous construction fails if the number of sides of the inscribed regular figure is *more than double* that of the given figure, for then it will be found that the first line *EB* does not cut the given figure. In that case the

middle point of a side must be taken for centre of similitude. It must be noticed, however, that the following construction can be applied to problems included in the previous case, *i.e.*, when the number of sides of the inscribed regular figure is *less than double* that of the given figure. It will be well to repeat the problem of inscribing a regular hexagon in a square by means of the construction indicated below.

4. *To inscribe a Regular Heptagon in an Equilateral Triangle.*



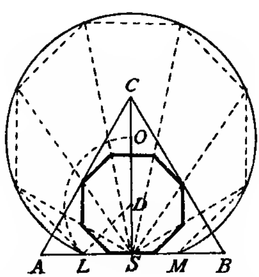
Take S, the middle point of AB, for Centre of Similitude. On CS, the perpendicular of the given triangle ABC, mark off CR equal to AB. Draw TR parallel to AB to meet AC in T. Then (*see* Chap. III., Prob. 7 (a), First Part, of which this construction is a modification) CT is the radius of the circle circumscrib-

ing the heptagon on *AB*. Draw the heptagon (*AO* = *CT*), and join the points *E*, *F*, etc., to *S*. Let *SF* cut *BC* in *f*. Draw *fe* parallel to *FE*.

Then *fe* is a side of the regular heptagon, which can be stepped out.

The construction with the middle point of a side for Centre of Similitude has an additional advantage. The whole of the side, on which it lies, need not be taken for the line on which to draw a figure similar to the figure to be inscribed. Instead, a line may be taken by marking off equal distances on either side of the Centre of Similitude. By this means an otherwise inconveniently large figure may be reduced to convenient dimensions; as follows:—

5. *To inscribe a Regular Octagon in an Equilateral Triangle of 3″ side, having one side on the base of the Triangle.*

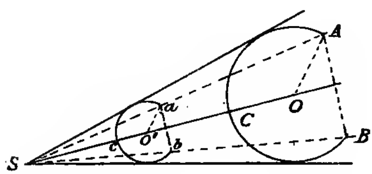


From *S*, the middle point of *AB*, mark off, on *AB*, equal lengths *SL*, *SM*, each about ·6″ long. On *LM* describe a regular octagon, and proceed as before.

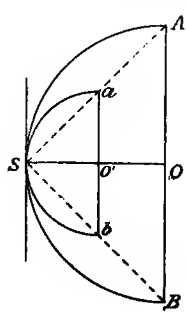
CENTRES OF SIMILITUDE OF CIRCLES AND ARCS OF CIRCLES.

The Centres of Similitude of two circles are the intersections of their common tangents.

Also, if ABC, abc, be two similar segments of circles with parallel chords, AB, ab, the lines joining Aa, Bb, will pass through the Centre of Similitude S. So also will the line joining O, O', the two centres; while OA, $O'a$ will be parallel.

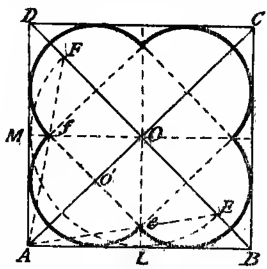


If the arcs are to touch a given straight line or circle at a given point, then the point of contact represents the intersection of common tangents, and is itself the Centre of Similitude.



The three following problems will serve to show the general principles of Centres of Similitude as applied to arcs of circles inscribed in regular figures:—

6. *To inscribe four continuous Semicircles in a Square, each touching two sides of the Square, and with the four diameters forming a second Square.*

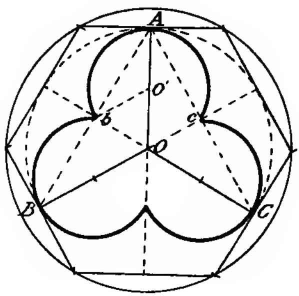


ABCD is the square, *O* the intersection of the diagonals.

One of the circles is to touch *AB*, *AD*; therefore *A* is its centre of similitude. Also, by symmetry with the other three semicircles, its diameter must be parallel to *BD*, and with the extremities of this diameter on *OL*, *OM*. Therefore, we must draw some semicircle to touch *AB*, *AD*, having its diameter in the same direction as *BD*. For simplicity, draw the semicircle *ELMF*, of which *O* is the centre. Join *AE*, *AF*, cutting *OL*, *OM* in *e*, *f* respectively.

Then *ef* is the diameter of the required semicircle, which can be now drawn, as also the other three semicircles.

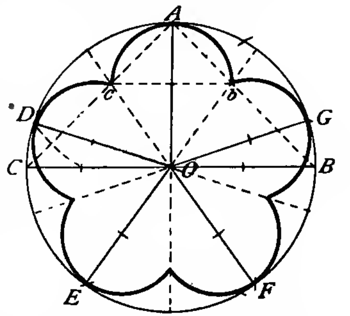
7. *In a Regular Hexagon inscribe three equal continuous circular Arcs, each equal to two-thirds of a complete Circle, and touching alternate sides of the Hexagon.*



A, B, C, the three middle points of alternate sides, are the points at which the arcs must touch, and are therefore Centres of Similitude. Also the arc touching at A must, by symmetry, terminate on BO, CO produced.

If O be taken for centre and OA for radius, then the arc BAC is two-thirds of a circle. Join BA, CA, cutting CO, BO produced in b, c respectively. Then b, c are the extremities of the arc required. The centre O' is found by drawing bO' parallel to BO. The other centres are found by stepping out OO', from O, on the lines OB, OC.

8. *To inscribe five continuous Semicircles in a given Circle, each of the Semicircles touching the Circle.*



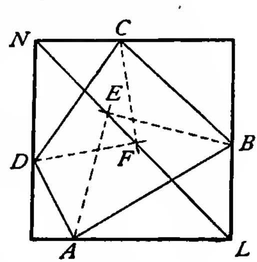
A, D, E, F, G are the points of a regular pentagon inscribed in the circle, and are the points at which the semicircles must touch the circle.

Thus A is the Centre of Similitude for the semicircle which touches at A, and, by symmetry, the extremities of its diameter must lie on EO, FO produced.

Now BAC is a semicircle touching the tangent at A. Therefore if BA, CA be joined, cutting OE, OF produced in b, c respectively, bc is the diameter of the semicircle which touches at A.

The following constructions for *describing* a square *about* a figure may be added here:—

9. *To describe a Square about a Quadrilateral.*



On AB, CD, opposite sides of the quadrilateral, draw isosceles right-angled triangles, with their vertices E, F towards one another. Join E, F. Then EF is the direction of a diagonal of the square, which may be constructed by drawing lines through A, B, C, D, making angles of 45° with EF.

Note.—It will be noticed that the whole of this figure can be drawn with set squares only.

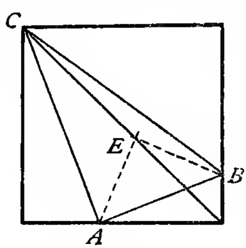
Proof. Modify the construction by saying, draw AL, DN, each making an angle of 45° with EF, and join BL, CN.

$$\text{Then } \angle ALE = 45° = \angle ABE.$$
$$\therefore \quad A, L, B, E \text{ are concyclic.}$$
$$\therefore \angle BLE = \angle BAE = 45°.$$
$$\therefore \angle ALE = \angle BLE, \text{ and each is } 45°.$$
$$\therefore \angle LFE \text{ is a diagonal of a square.}$$

Similarly NEF is a diagonal of a square, and these two lines being coincident they form the diagonal of the same square.

This construction is only true if EF cuts the lines AB, CD on which the isosceles triangles are drawn.

10. *To describe a Square about a Triangle.*



The construction is practically the same as in Problem 9, if C, D are considered as coincident points.

On AB, one of the sides of the triangle ABC, draw the right-angled isosceles triangle ABE.

Join C, E. Then CE is the direction of a diagonal of the square.

Note.—Three squares can be drawn about the triangle.

EXERCISES—IV.

(In every example measure the side or radius constructed.)

1. Inscribe a regular hexagon in an equilateral triangle of 2″ side.
2. Inscribe a regular octagon in a square of 2″ side.
3. Inscribe a regular dodecagon in a regular hexagon of $1\frac{1}{2}''$ side.
4. Draw a regular nonagon, and in it inscribe another, concentric with it, having its side 1″ long.
5. Inscribe in equilateral triangles of $1\frac{1}{2}''$ side the following regular figures:—

    (1) and (2) A pentagon in two different ways.

    (3) An octagon.
6. In a square of $1\frac{1}{2}''$ side inscribe a regular heptagon, using the middle point of a side for centre of similitude.
7. In a regular hexagon of $1\frac{1}{2}''$ side inscribe a regular heptagon.
8. In an equilateral triangle of 3″ side inscribe a regular nonagon.
9. In a square of 2″ side inscribe a semicircle touching two adjacent sides and having its diameter parallel to a diagonal of the square.

10. In an equilateral triangle of 3″ side inscribe three continuous semicircles, each to touch two sides of the triangle.

11. In an equilateral triangle of 3″ side inscribe three continuous semicircles, each to touch one side of the triangle.

12. In a square of $2\frac{1}{2}''$ side inscribe four continuous semicircles, each to touch one side of the square.

13. Inscribe seven continuous semicircles in a heptagon inscribed in a circle of $1\frac{1}{2}''$ radius, each touching two sides of the heptagon.

14. In a square of 2″ side inscribe two arcs, each three-quarters of the circumference of the same circle, to touch opposite sides of the square at their middle points.

15. In a regular octagon of 1″ side inscribe four equal arcs, each three-quarters of a circumference of a circle, to touch alternate sides of the octagon.

16. In a circle of $1\frac{1}{2}''$ radius inscribe seven continuous semicircles, each touching the given circle.

17. In a circle of $1\frac{1}{2}''$ radius inscribe three continuous arcs, each five-sixths of a circumference of a circle, and each touching the given circle.

18. *AB*, *BC*, *CD* are three consecutive sides of a regular hexagon of $1\frac{1}{2}''$ side. About the quadrilateral *ABCD* describe a square.

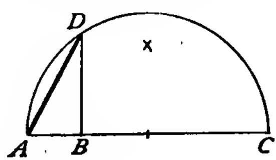
19. About an equilateral triangle of 2″ side describe a square.

CHAPTER V.

PROPORTION.

Note on Mean Proportional.—It has been shown in Chapter v., First Part, if ADC is a semicircle and DB perpendicular to AC, that DB is a mean proportional between AB and CB, and that

AD is a mean proportional between AC and AB.

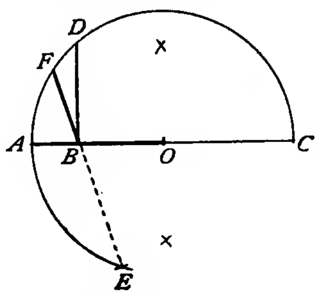


It is to be noticed that the former construction is to be used when the two lines, between which the mean proportional is to be found, are measured *in opposite directions* from a common point, *i.e.*, BA, BC; while the latter is more convenient when the two lines both start from a common point *in the same direction*, *i.e.*, AC, AB.

The latter method is also to be used if the line AC would be inconveniently long, when the lengths are measured in opposite directions, *e.g.*, a mean proportional between 6″ and 5″.

Problems on Proportion.

1. *To find the Arithmetic, Geometric, and Harmonic Means between two given lines a, b.*



Measure AB, BC, equal to a, b respectively, *in opposite directions from B* along a straight line.

Bisect AC at O. Then OA is the Arithmetic Mean between a, b.

Find BD, the Mean Proportional between AB and BC. Then BD is the Geometrical Mean between a, b.

With centre B and radius OA draw an arc to cut the circle of which O is the centre and OA radius at E. Join EB, and produce it to meet the circle at F. Then BF is the Harmonic Mean between a, b.

PROOF. From the Algebraic definitions of the Means

$$A = \text{Arithmetic Mean} = \frac{a+b}{2} = OA,$$

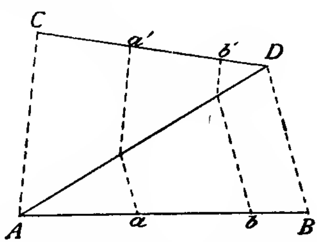
$$G = \text{Geometric Mean} = \sqrt{ab} = BD,$$

while, if H = Harmonic Mean, then $AH = G^2$.

Now if BD be produced to meet the circumference at D',

$$FB \cdot BE = DB \cdot D'B \qquad \text{(Euc. III. 35.)}$$
$$= DB^2,$$
$$\text{or } FB \cdot A = G^2,$$
$$\text{that is } FB = H.$$

2. *To divide a given finite straight line similarly to a given divided line, when the two straight lines have not a common extremity.*



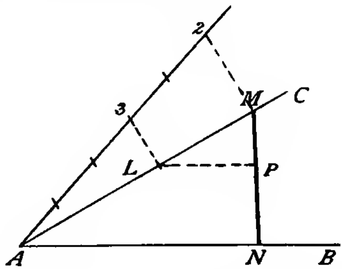
AB, CD are the two given lines, and AB is divided at a, b, etc.

Join AD, and divide it similarly to AB. (*See* Chap. V., First Part.)

Then, by a similar construction, divide CD similarly to AD.

PROOF. Euc. VI. 4.

3. *To draw a straight line through a given point so that the parts intercepted by two given straight lines are in a given ratio (say 3 : 2).*



Through P, the given point, draw PL parallel to AB, one of the given straight lines AB, AC.

On AC find M, so that $AL : LM$ is in the given ratio. Join MP, cutting AB at N.

Then MN is the required straight line.

PROOF. Euc. VI. 4. The construction is the same if P is outside the angle BAC.

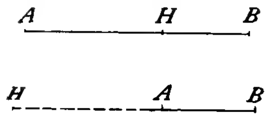
Note.—See Chapter I., Problem 6.

Extreme and Mean Ratio or Medial Section—Angle of 18°.

DEFINITION. A line AB is divided at H into Extreme and Mean Ratio, or Medial Section, when

$$AB : AH :: AH : BH,$$
$$\text{or } AB \cdot BH = AH^2.$$

(*See* Euclid II. 11.)



The straight line AB can be divided both internally and externally at H to satisfy the above condition, as in Problems 4, 5 below.

Now if $AB = a$ and $AH = x$, then, for internal medial section, we have

$$AB \cdot BH = AH^2,$$

$$\text{or } a\,(a - x) = x^2,$$

a quadratic equation, from which it follows that

$$\frac{x}{a} = \frac{\sqrt{5} - 1}{2}$$

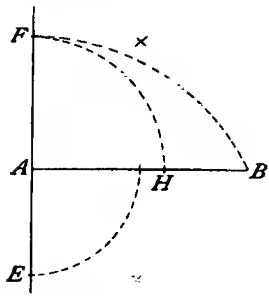
$$\text{and hence that } \frac{AH}{2 \cdot AB} = \frac{\sqrt{5} - 1}{4}.$$

Now the ratio on the right hand side is the value of sin 18°, so that it is possible to use the construction for Medial Section for constructing an angle of 18°. (*See* Problem 6.)

The following construction for Medial Section is preferable to that given in Chapter v., Problem 8, First Part, for it can be used for external as well as internal section, and is more in accordance with the construction of Euclid II. 11, from which the proof is derived.

Problems on Medial Section.

4. *To divide a given straight line internally in Medial Section.*



On the perpendicular through A to the given straight line AB mark off AE equal to half AB.

With E for centre, and EB for radius, draw an arc cutting EA at F in EA *produced through* A.

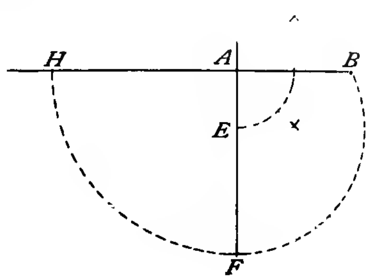
With A for centre, and AF for radius, draw an arc to cut AB at H.

Then AB is divided in Medial Section at H, so that

$$AB \cdot BH = AH^2.$$

Proof. Euclid II. 11.

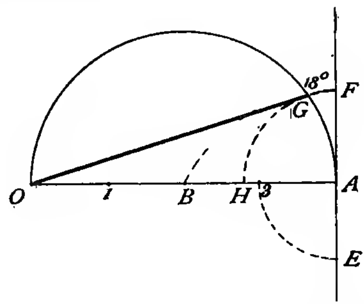
5. *To divide a given straight line externally in Medial Section.*



The construction is identical with that of the previous problem, except that the arc drawn with E for centre cuts AE *produced through* E, while H is on BA *produced through* A.

Angle of 18°.

6. *To construct an angle of 18° at a given point in a given straight line.*



If O is the given point in the line OA, step out any convenient distance $O1$ four times, giving B as the second point, A as the fourth. Through A draw a perpendicular to OA, and on it mark AE equal to $A3$.

With E for centre and radius EB draw an arc to cut EA produced in F. With A for centre and AF for radius draw an arc to cut the semicircle described on OA in G. Join OG.

$$\text{Then } \llcorner AOG = 18°.$$

PROOF. The construction has divided AB at H in Medial Section,

so that $AB \cdot BH = AH^2$

$$\text{and } \therefore \quad \frac{AH}{AB} = \frac{\sqrt{5}-1}{2} \text{ (see p. 45).}$$

$$\therefore \text{ since } AG = AH, \frac{AG}{OA} = \frac{\sqrt{5}-1}{4}$$

$$= \sin 18°, \text{ by Trigonometry;}$$

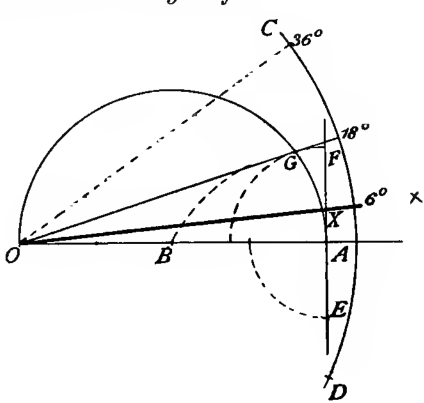
also $\llcorner\, OGA = 90°$, $\therefore \llcorner\, AOG = 18°$.

Note.—By applying the construction for dividing AB *externally* in Medial Section we could construct in the same way an angle of 54°. It is, however, just as simple to construct 18° first and then step out, three times, the arc cut off by OA, OG on a circle of which O is the centre.

Angles Depending on the Construction of 18°.

By combining the construction given in Problem 6, for 18° and its multiples, with that given in the First Part (Chapter I.) for 15° and its multiples, we can construct all angles which are multiples of 3°, that is of 18° − 15°. For example,

7. *To construct an angle of 6°.*



Construct AOG equal to 18°, as in Problem 6; hence its double, AOC, marked off on the arc of which O is centre.

Now construct, towards OA, the angle COX, equal to 30°.

$$\text{Then } \angle AOX = 36° - 30°$$
$$= 6°.$$

EXERCISES—V.

1. Find, to two places of decimals, the mean proportional between 5 and 6.
2. Find the harmonic mean between 1″ and 3″. Measure it and test by Arithmetic.
3. Prove geometrically that $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$.
4. Prove geometrically that $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.
5. Draw two parallel straight lines 2″ apart, AB to be 2″ and CD $2\frac{1}{2}$″ long. Divide CD at K, L so that

   $$CK : KL : LD :: \tfrac{1}{3} : \tfrac{1}{6} : \tfrac{1}{2}.$$

   Divide AB similarly to CD, and show that one point of division bisects AB.
6. Draw $AB = 2''$, and $CD = 3''$, parallel and 1″ apart. Trisect AB at the points 1, 2. On AB mark AK, AL, mean proportionals between $A1$, AB, and $A2$, AB respectively. Divide CD at K', L' so that

   $$CK' : K'L' : L'D :: AK : KL : LB.$$

   Write down the length of CK'.

7. Draw two straight lines AB, AC inclined at an angle of 45°. Take P, on the bisector of the angle BAC, 2″ from A. Draw MN through P and terminated by AB, AC so that P is a point of trisection of MN. Test the result by trisecting MN.

8. Divide a line 5″ long internally in Medial Section. Measure the parts.

9. Divide a line 2″ long externally in Medial Section. Measure the part produced.

10. Construct angles of 18° and 36°.

11. Construct an angle of 54°.

12. Divide a right angle into five equal parts.

13. At a given point in a given straight line construct an angle of 9°.

14. Construct an angle of 63°.

15. Construct an angle of 27°.

CHAPTER VI.

SCALES AND IRREGULAR FIGURES.

WHEN the primary and secondary divisions have been found on the base line of a scale, the Marquois Scales can be conveniently used for drawing the parallels to the base line. The paper should be reversed before the parallels are ruled, so that it may be seen how long the lines ought to be and that all are parallel. In the case of a Diagonal Scale it is well to draw the two perpendiculars to the base line at the ends, and only these, before reversing and drawing the parallels; but for a Plain Scale this is unnecessary.

When the parallels to the base line have been drawn, and before the ruler is removed from its position parallel to them, the perpendiculars should be drawn at once with a set square working on the ruler as straight edge.

The distance between the parallels may vary according to the length of the scale. For a Plain Scale of ordinary length (5 to 6 inches) the parallels should be about one-eighth or one-sixth `f an inch apart. For this purpose

use the 40 or 60 Marquois Scale, and move through five or ten divisions respectively at a time. A Diagonal Scale should not, in general, exceed one inch in breadth. Thus for eight parallels use the 50 scale, and move through five divisions at a time; for ten or twelve parallels use the 60 scale at intervals of five divisions. The 60 scale is a convenient one to use, because the four lines required for printing can be drawn at the same time as the parallels. The intervals for these, as for all printing, will be $\frac{3}{60}$ths, $\frac{4}{60}$ths, and $\frac{3}{60}$ths respectively.

In a Diagonal Scale in which there are not separate names for the secondary and diagonal divisions care must be taken to have the right number of each. For instance, if the primary division is one mile, and chains are to be shown, 80 chains to the mile; or if the primary division is a furlong, and yards are to be shown, 220 yards to the furlong. It must be remembered that it is most important to be able to read off lengths direct from any scale, without performing mental sums in addition or subtraction, and that this can only be done, in such examples as those given above, by making each secondary division represent 10 units. Thus, for 80 chains there will be 8 secondary divisions and 10 parallels; and for 220 yards, 22 secondary divisions and 10 parallels.

With regard to primary divisions, it must be carefully borne in mind that, if the total length is between 10 and 100 units, it will, with one exception, always represent

an exact multiple of 10, and that each primary division will represent 10 units. If the total length is between 100 and 1,000, each primary division represents 100, and so on. The single exception is when the total length has an odd 50 or 500, for instance, 250 or 3,500. In this case the primary divisions may be each 50 or 500 respectively.

The two following examples of Scales are given as types which may be followed.

If the length is determined by Rule of Three, the following rules will always hold:—

1. The length whose equivalent number of inches is to be determined is always in the second term. The actual length given is always in the first term, and the number of inches belonging to it in the third.

2. If, now, in a Comparative Scale the first and second terms are not in the same denomination, make them so by means of the relations given in the question. The required length can now be found as in Ex. 2, p. 55.

3. If the representative fraction is given; then, as before, the length to be determined is in the second term; the denominator of the R.F. is in the first term; while, since the first term of the proportion is in the same denomination as the second, the numerator of the R.F. in the third term will also be of this denomination and must be brought to inches. (*See* Ex. 1, p. 54.)

Ex. 1. *Draw a scale of furlongs and chains for a sketch, whose R.F. is* $\frac{1}{8000}$. *Show* 5 *furlongs.*

[1 *furlong* = 220 *yards.* 1 *chain* = 22 *yards.*]

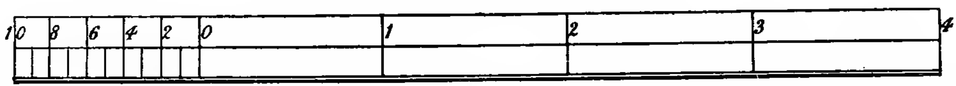
$$\overset{\text{Fur.}}{8000} : \overset{\text{Fur.}}{5} :: \overset{\text{Inches.}}{1 \times 220 \times 36}$$

$$\therefore 5 \text{ furlongs} = \frac{1}{\underset{\underset{80}{\cancel{400}}}{\cancel{8000}}} \times \frac{\cancel{5}}{1} \times \frac{\overset{11}{\cancel{220}} \times 36}{1}$$

$$= \frac{396}{80}$$

$$= 4{\cdot}95''. \quad \text{Also 1 furlong} = 10 \text{ chains.}$$

Scale of Furlongs and Chains, R.F. $\frac{1}{8000}$.



Note.—The parallels are drawn with the 60 Marquois Scales, 10 divisions apart. The four lines for printing begin 10 divisions above. Neither these, nor the pencil lines used in dividing for the primary divisions, need be rubbed out in Examination or on ordinary occasions.

Ex. 2. *On a scale of 3·85″ to the mile, construct a diagonal scale of Russian Versts, showing smallest divisions of 10 sagenes. Find the R.F.* [1 *verst* = 500 *sagenes*, 1 *sagene* = 7 *English feet.*]

$$\overset{\text{Feet.}}{1 \times 5280} : \overset{\text{Feet.}}{1 \times 500 \times 7} :: 3{\cdot}85''$$

$$\therefore 1 \text{ verst} = \frac{1}{\underset{1056}{\cancel{5280}}} \times \frac{\overset{100}{\cancel{500}} \times 7}{1} \times \frac{3{\cdot}85}{1}$$

$$= \frac{2695}{1056}$$

$$= \frac{336{\cdot}875}{132}$$

$$= \frac{84{\cdot}218 \text{ - -}}{33}$$

$$= \frac{28{\cdot}072 \text{ - - -}}{11} = 2{\cdot}55{,}2'' \text{ - - -}$$

$\therefore$ 2 versts = 5·10,4″ (*See* p. 56, note 3.)
= 5·10″ nearly.

$$\text{R.F.} = \frac{3{\cdot}85}{63360}$$

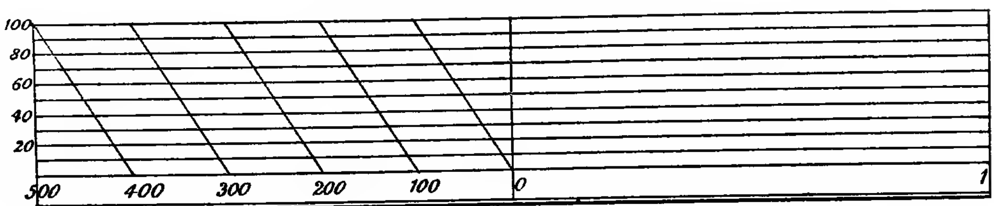
$$= \frac{385}{6336000}$$

$$= \frac{77}{1267200}$$

$$= \frac{11}{181028{\cdot}4 \text{ - - -}}$$

$$= \frac{1}{16457{\cdot}1 \text{ - - -}} = \frac{1}{16457} \text{ nearly.}$$

Scale of Versts and Sagenes, R.F. $\frac{1}{16457}$.



EXERCISES—VI.

Scales.

Note 1.—In each example all the calculations must be clearly shown on the paper.

The Representative Fraction must be calculated in each scale in which it is not given.

Note 2.—The R.F. must be calculated from the data given in the question. If it were calculated from the determined length of the scale, which is generally only approximate, the result might differ considerably from the real value.

Note 3.—If the greatest length to be shown on the scale is not stated, then first find the scale length of the greatest unit to be measured and draw a scale to represent as many of these units as will give a length of scale of about 5 to 7 inches. (*See* Ex. 2, p. 55.)

1. On a map 13·2 miles are represented by 6·6″. Draw a plain scale for the map to show 10 miles and quarters of a mile. Show all your calculations. Find the representative fraction, and show by two small circles the points you would take in order to measure $7\frac{3}{4}$ miles.

2. Draw a diagonal scale to show 10 yards, feet, and inches, having given that 10 miles = 1 furlong.

3. Draw, by the diagonal method, to read miles and furlongs up to 30 miles, a scale in which

$$\text{R.F.} = \frac{1}{360000}.$$

4. On a map 10 kilometres are denoted by 1·94″. Draw a plain scale of 10 miles.

   [1 kilometre = 1,000 metres. 1 metre = 39·37″.]

5. On a map 67·5 miles are represented by 9·3 inches. Draw a scale of miles for the map showing 30 miles. Show furlongs by the diagonal method. Show by two small circles on the scale the points you would take in order to measure off a distance of 17 miles 3 furlongs.

[1 mile = 8 furlongs = 1,760 yards.]

6. Draw a scale showing all distances from 4 furlongs to single poles for a plan in which 5·94 inches represent 1,000 yards.

[1 furlong = 40 poles = 220 yards.]

7. On the scale of 36 yards to 25 miles draw a diagonal scale to read yards and perches up to 1 furlong, given

1 furlong = 40 perches.
1 perch = $5\frac{1}{2}$ yards.

8. To the same scale draw a plain comparative scale of Irish perches, to show distances from 1 perch up to 1 furlong, given that

1 mile = 8 furlongs.
1 Irish mile = 1·273 English miles.

9. Draw a scale of miles, furlongs, and chains when 11″ represent 8 miles.

[1 mile = 8 furlongs. 1 chain = 22 yards.]

10. Draw a scale of linear chains and yards for a plan on which 1 acre is represented by 10 square inches.

[1 acre = 4,840 square yards. 1 chain = 22 yards.]

11. For the same plan as that of the previous question draw a scale to show poles and yards. Show by two small circles the distance 18 poles 3 yards.

[$5\frac{1}{2}$ yards = 1 pole.]

12. Draw a scale of quarters and lbs., when 1 cwt. is represented by 6 inches.

[1 cwt. = 4 qrs. = 112 lbs.]

13. If 20 months can be represented by a line 4 feet 8 inches long, draw a scale to read months, weeks, and days.

[1 month = 4 weeks. 1 week = 7 days.]

14. A body of troops takes 1 hour to march $2\frac{1}{2}$ miles. Construct a time scale to show intervals of five minutes. The scale of the map is 2 inches to the mile.

15. On a scale of 7·2 yards to 25 miles draw a scale so that the greatest distance to be read is 4 furlongs and the smallest 2 yards. Show all calculations and find the R.F.

[1 mile = 8 furlongs. 1 furlong = 220 yards.]

16. An English map is drawn to a scale of 6 inches = 1 mile. Construct a scale for use with this map in French measure. One metre = 39·37 inches (nearly). Give the Representative Fraction, and indicate on the scale a distance of 1,470 metres.

17. In a certain country 10 units of area equal an acre. Make a scale for use in this country which will read to tenths and hundredths on the diagonal part.

The Representative Fraction is $\frac{1}{528}$.

Indicate by two marks on the scale 3·78 units of length.

18. The distance on a sketch was found to be wrong owing to an error in pacing. On comparing the sketch with a correct survey, it was found that 685 yards on the

survey are equal to 650 yards on the sketch. It was intended that the sketch should be drawn on a scale of six inches to the mile. Draw a scale of yards to suit the sketch. Show 1,000 yards as measured on sketch.

19. The distances measured on a plan do not agree with those taken from an accurate Ordnance survey. Thus a length of 690 yards on the Ordnance survey is represented by only 655 yards on the plan given, and other distances are diminished in a like ratio. The plan was supposed to have been executed on a scale of six inches to the mile. Draw a true scale for the plan, and give its Representative Fraction.

Irregular Figures.

20. A road goes straight for 2 miles, then turns through an angle of 60° to the left for a distance of 1 mile, then through an angle of 120° to the right for 2 miles, and finally through an angle of 60° to the right for one more mile. Draw a plan of the road on a scale of $1\frac{1}{2}''$ to the mile, and measure in miles and quarters of a mile the direct distance between the extreme points.

The scale required must be drawn, and the R.F. written above it.

21. Draw a semicircle of which O is the centre, and on a scale, which need not be drawn, of 1 inch to the foot draw the figure $ABCDE$, having given

$$OA = 2 \text{ feet}$$

$\angle AOB = 30°$	$OB = 2{\cdot}2$ feet
$\angle AOC = 105°$	$OC = 2{\cdot}6$ feet
$\angle AOD = 135°$	$OD = 1{\cdot}4$ feet
$\angle AOE = 180°$	$OE = 2{\cdot}5$ feet

Find also the area of the figure in square feet.

22. (1) A distance of ·041 mile being represented by 110″, show that a scale of 3 metres will be 5″ long. Draw the scale to read metres and decimetres, and find the R.F. [1 metre = 39·36″.]

(2) Using the same scale, draw a figure $ABCD$, given that AB = 2·4 metres, and that

$$\angle ABC = 75^\circ \qquad \angle BAC = 30^\circ$$
$$\angle ABD = 45^\circ \qquad \angle BAD = 60^\circ$$

The angles are to be found by construction.

Write down in metres the length of CD. Reduce the quadrilateral to a triangle equal to it, and hence find its area in square metres.

23. Draw the figure $ABCDE$, having given that

$$AB = 2{\cdot}8'', \quad BC = 1{\cdot}65'', \ CD = 2{\cdot}80''$$
$$DE = 1{\cdot}50'', \ BD = 2{\cdot}7'', \quad AD = 3{\cdot}25''$$
$$\angle CDE = 110^\circ$$

Measure and write down the length of AE.

24. A point A is taken in line with two inaccessible points P and Q (P being the nearer) which are known to be 200 yards apart. A line AB, 300 yards long, is set off at right angles to APQ, and the angle ABQ is found to be $67\frac{1}{2}^\circ$. Determine by a geometrical construction the distance from P to A, and measure with a protractor the magnitude of the angle ABP. The angle ABQ should be constructed.

Scale (which is to be properly drawn and figured), 150 yards to an inch, showing 1,000 yards.

25. A vertical pole 100 feet high casts a shadow on the ground, the sun's rays being inclined to the horizontal at an angle of 38°.

If the pole is represented by a line 3″ long, construct a scale to measure the shadow, and find its length in feet by means of the scale.

CHAPTER VII.

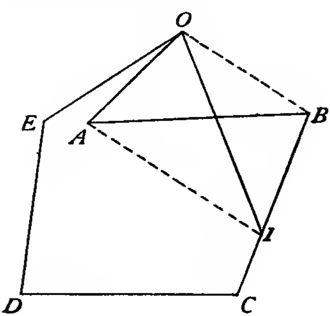
THE DETERMINATION OF THE AREA OF A RECTILINEAR FIGURE.

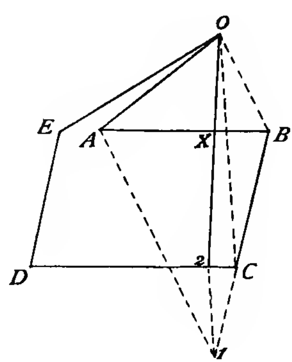
THE construction given in Chapter VII., Problems 6 and 7, First Part, for the reduction of a rectilinear figure to an equivalent triangle of equal area, having its vertex on a side of the figure, can be extended so that the vertex of the triangle is a point either within or outside the given figure.

Problems.

We shall first consider the *case of failure* which sometimes occurs when the figure has a re-entrant angle (*e.g.*, the angle OAB in the figures below).

1. *To reduce a given Rectilinear Figure with a re-entrant angle to an equivalent triangle, and hence find the area.*





OABCDE is the given rectilinear figure, having the re-entrant angle *OAB*.

Proceeding as on page 70, First Part, take *OA*, *AB* for the two first sides, and *OB* for the corresponding diagonal, draw a parallel to *OB* from *A* to meet the third side, *BC*, at 1.

First, if 1 is between *B* and *C* the triangle *AB*1 can be replaced by *AO*1, and the construction of page 70 can be applied throughout.

Second, if 1 is *on BC produced* the triangle *AB*1 cannot be replaced by *AO*1, for *AB*1 is not a part of the original figure. Thus the construction fails. Then through 1 draw 12 parallel to *OC*, the second diagonal through *O*, to meet the fourth side *CD* at 2, and join *O*2.

If *O*2 meets *AB* at *X*, then it can be proved that *XBC*2 is equal to, and can therefore be replaced by, *AOX*.

Thus the given figure is equivalent to *O*2*DE*, which can be reduced to an equivalent triangle in the ordinary way, having *O* for the vertex and base in *CD*.

Proof. $\triangle\ O2C = \triangle\ O1C$. (Euc. I. 37.)

Add to each $\triangle\ OBC$

$\therefore\ O2CB = \triangle\ O1B$

$= \triangle\ OAB$. (Euc. I. 37.)

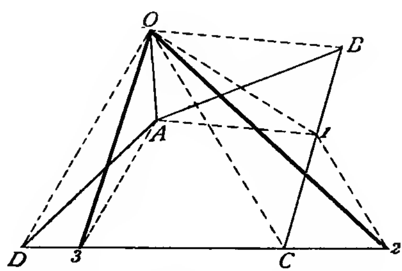
Take away the common part, the $\triangle\ OXB$,

$\therefore\ X2CB = \triangle\ XAO$.

Note.—Compare the construction of Chapter x., Problem 6 (First Part).

Now suppose, in the previous figures, that A and E coincide. Then the area of the figure will be that of $ABCD$; but the figure can still be reduced to an equivalent triangle, having its vertex at O and its base in CD; for we can, without change of area, consider $ABCD$ to be a figure having OA for its first and AO for its last side. Hence the following problem:—

2. *To reduce a given Rectilinear Figure to an equivalent triangle, having its vertex at a point inside or outside the figure, and its base in one of the sides of the figure.*

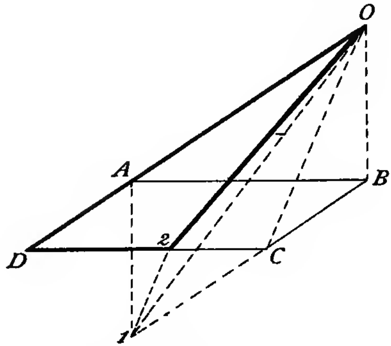


Join O, the given point, to A, one of the points of the given figure $ABCD$.

Considering OA, AB as the two first sides and OB as the first diagonal, proceed to reduce the figure to an equivalent triangle, with the modification, if necessary, of the previous problem.

The first figure shows the reduction to an equivalent triangle when no modification is required.

In the second figure—a parallelogram reduced to an



equivalent triangle having its vertex on one side produced—the line $O1$ cuts the side CD before it meets BC, so that 12 must be drawn parallel to OC, as explained in Problem 1.

The construction when O is inside the figure is exactly the same, and in general presents no difficulty.

Note on the Area of a Regular Figure.

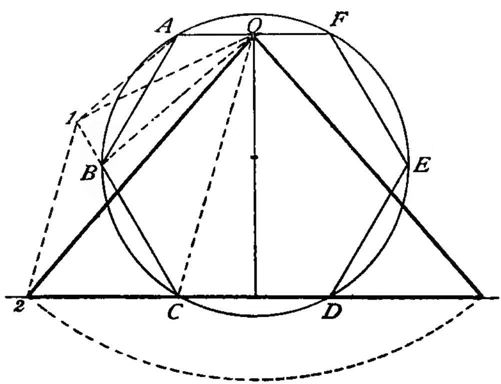
If the area of a regular figure is to be found, and no restriction is placed on the positions of the vertex and base of the equivalent triangle, two points are to be noticed.

(1) The point chosen for the vertex of the triangle should be on a side of the regular figure, and should be such that the perpendicular from it on to the base line should be as large as possible. For since

$$\text{area of triangle} = \tfrac{1}{2} \times \text{base} \times \text{perpendicular},$$

if the perpendicular is made large, the base will be correspondingly small.

(2) If the regular figure has an *odd* number of sides, one of the angular points should be chosen for vertex; while, if the number of sides is *even*, the middle point of a side should be taken for vertex. In both cases the opposite side of the regular figure will be taken for the base line of the equivalent triangle; and, since the perpendicular from the vertex on to the base line divides the regular figure into two equal symmetrical parts, the reduction need only be made on one side of the perpendicular.



The figure shows the construction for finding the area of a regular hexagon.

EXERCISES—VII.

1. Draw a regular pentagon of 2″ side. Reduce it to a triangle of equal area, having its vertex at one of the angular points of the pentagon and its base in the opposite side. Determine the area.

2. Repeat the regular pentagon of the previous question by pricking the angular points through on to another piece of paper and joining up. Reduce the pentagon to an equivalent triangle, having its vertex at the intersection of two diagonals and its base in one of the sides farthest away. Determine the area.

3. Reduce a regular hexagon of $1\frac{1}{2}''$ side to a triangle of equal area with its vertex at the middle point of one of the sides of the hexagon. Determine the area.

4. Repeat the previous hexagon and find its area by taking one of the angular points for vertex of the equivalent triangle.

5. Draw a regular heptagon of $1\frac{1}{2}''$ side, reduce it to an equivalent triangle, and hence find the area.

6. Repeat the previous heptagon, and find its area by reducing to an equivalent triangle whose vertex is at the intersection of a shorter and longer diagonal.

7. Draw the square $ABCD$ with a side of $2\frac{1}{2}''$. Produce DA to O so that $OA = \frac{1}{2}\,AC$. Reduce the square to an equivalent triangle with O for vertex and base in CD. Show from the triangle that the area is 6·25 square inches.

8. Draw a rectangle 3·6″ by 1″. At the middle point of one of the longer sides erect a perpendicular 3″ long and cutting the opposite side. Reduce the rectangle to a triangle of equal area, having this perpendicular for altitude and base in the bisected side. Show that the area of the triangle is 3·6 square inches.

9. Describe a regular decagon on a line 1″ long. Bisect it by a line joining two of its opposite angular points A, B. Draw a perpendicular to AB through the centre of the circumscribing circle. Let this perpendicular cut the decagon at O. With O for vertex, reduce the half of the decagon on which O does not lie to an equivalent triangle, and find the area of this triangle.

10. On a line DE, 1″ long, and on the same side of it draw a regular heptagon and a regular dodecagon. Let the line joining the centres of the two circumscribing circles cut the dodecagon at O. With O for vertex and DE for base line reduce half the area included between the two regular figures to an equivalent triangle and find its area.

CHAPTER VIII.

SURDS AND SUMS AND DIFFERENCES OF SQUARES.

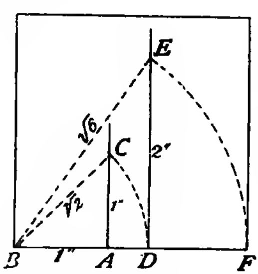
THE following method of treating surds and kindred problems fulfils the requirements of the principle that, in the problems of geometrical drawing, the constructions adopted should, as far as possible, be clearly indicated in the figures themselves.

It is to be noticed that any length may be selected for unit; but the advantage of using 1″ for unit is that, then, the measured length in inches exactly represents the root required.

The following are typical problems.

Problems.

1. *To draw a square of 6 inches area.*



Here we see that

$$6 = (\sqrt{2})^2 + 2^2.$$

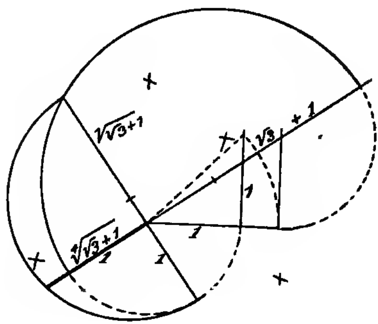
The dotted circles show that

$$BD = BC$$
$$= \sqrt{2}$$
and $$BF = BE$$
$$= \sqrt{6}.$$

The measured length of BF should be 2·45″.

Note.—This method of construction may be continued to any extent along the line *BF*. It is, too, readily accomplished with set squares. It is to be noticed that any of the perpendiculars, such as *DE*, can be drawn as easily *below* *BF* as above; and that they should be drawn alternately above and below if there is any danger of *BC*, *BE*, or any other pair, being nearly coincident when both are above *BF*.

2. *To find the value of* $\sqrt[4]{\sqrt{3}+1}$.

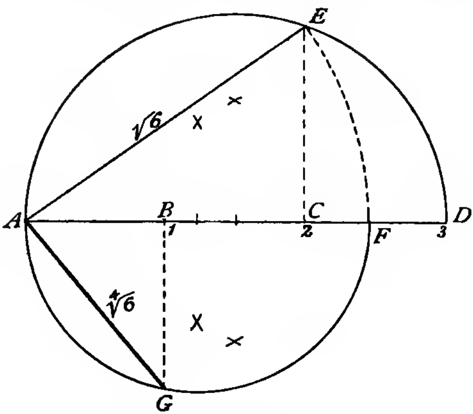


First, find $\sqrt{3}$ as in Problem 1, then $\sqrt{\sqrt{3}+1}$ as a mean proportional between $\sqrt{3}+1$ and 1. The dotted circles indicate that the right lengths have been taken.

Lastly, find a mean proportional between $\sqrt{\sqrt{3}+1}$ and 1 as in the figure.

The length should be 1·29″.

3. *To find the value of* $\sqrt[4]{6}$, *using* $1\frac{1}{2}''$ *for unit.*



Step out the unit $1\frac{1}{2}''$ three times along AD, so that

$$AB = BC = CD.$$

Draw the semicircle on AD and the perpendicular CE.

Then AE is the mean proportional between AC and AD, *i.e.*, is $\sqrt{2 \times 3}$ or $\sqrt{6}$.

Now, as in the figure, find by the same process the mean proportional AG between AF ($= AE$) and AB, that is between $\sqrt{6}$ and 1. Thus $AG = \sqrt[4]{6}$.

Its measured length should be $2{\cdot}35''$.

Note.—Two useful principles are illustrated in this example.

1. As indicated on page 42, it is preferable to measure AC, AD *in the same direction*, and take AE for their mean proportional, when their total length, if they were measured *in opposite directions*, would be inconveniently long (here $7{\cdot}5''$).

2. If a double construction has to be made in the same figure, it is often possible to use both sides of the initial line and so avoid confusion.

4. *To find the square root of a large number, e.g.,* $\sqrt{33}$.

In such a case the process of Prob. 1, by sums and differences of squares, is tediously long, while the direct application of the mean proportional method would involve a line too long for practical progress.

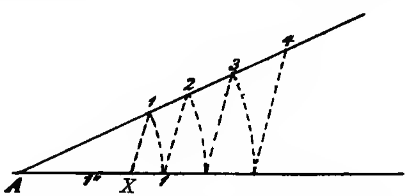
The latter method can, however, be modified, for

$$\sqrt{33} = \sqrt{11 \times 3}$$
$$= \sqrt{\frac{11}{2} \times 6}.$$

That is $\sqrt{33}$ is a mean proportional between 6 and $5\frac{1}{2}$. Its constructed length should be 5·74″.

Powers of Numbers.

5. *To find lines representing powers, such as the square, cube, fourth power, etc., of a given line.*



The *square* is the third proportional to 1—AX in the figure—and the given line, $A1$.

The *cube* is the third proportional to the line and its square.

The *fourth power* is the third proportional to the square and cube of the line; and so on.

The figure shows the determination up to $A4$, the fourth power of $A1$.

Note.—The greatest care is required in determining the higher powers.

PROOF.
$$AX : A1 :: A1 : A2$$
$$\therefore A2 = (A1)^2, \text{ since } AX = 1.$$
$$A1 : A2 :: A2 : A3$$
$$\therefore A1 . A3 = (A2)^2$$
$$= (A1)^4, \text{ or } A3 = (A1)^3$$
and similarly $$A4 = (A1)^4.$$

EXERCISES—VIII.

1. Draw a square equal to the sum of the squares on lines 1·1, 1·2, 1·3, and 1·4 inches long. Measure its side.

2. Construct a square equal to five times the square on a line ·9″ long. Measure its side.

Find, by the method of mean proportional, the value of the following :—

3. $\sqrt{18}$. 4. $\sqrt{30}$. 5. $\sqrt{23}$.

Using 1″ for unit, find the value of the following to two places of decimals :—

6. $\sqrt[4]{5}$. 7. $\sqrt[8]{\sqrt{3}+1}$. 8. $\sqrt[4]{\sqrt{3}+\sqrt{2}}$.

Using $1\frac{1}{4}$″ for unit, find the length in inches of the following to two places of decimals :—

9. $\sqrt[4]{3}$. 10. $\sqrt[8]{2}$. 11. $\sqrt[4]{\sqrt{2}+1}$.

12. Six equal circles of ·64″ radius are cut out of a circular sheet of lead of 2″ radius. The remainder is recast into a circular sheet of the same thickness. Find its radius.

Find graphically, to two places of decimals, the value of the following powers :—

13. $(1{\cdot}5)^3$. 14. $(1{\cdot}31)^4$. 15. $(1{\cdot}23)^5$.

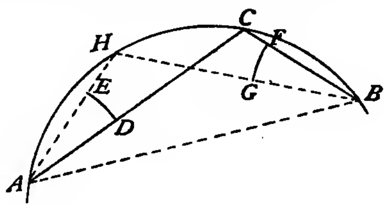
16. $(\sqrt{2})^3$. 17. $(\sqrt[4]{3})^5$. 18. $(\sqrt{5} - 1)^{\frac{5}{2}}$.

CHAPTER IX.

CIRCLES.

Problems.

1. *To find points on the circle passing through three given points without finding the centre.*



With two of the given points A, B for centres, and with some convenient radius, draw equal arcs DE, FG in *opposite* directions from the lines joining A, B to the third point C.

Join AE, BG, meeting at H.

Then H is on the circle.

PROOF. $\angle HAC = \angle HBC$.

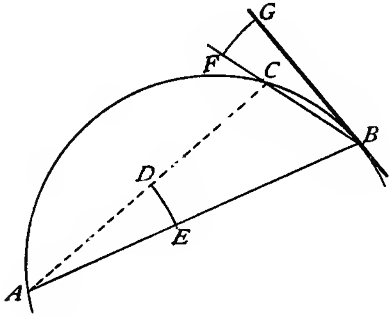
$\therefore$ H is on the circle. (Euc. III. 21.)

If H is on the opposite side of AB to C, the proof depends on Euc. III. 22.

COR. *Given a portion of a circle, find points on the part not given without finding the centre.*

This is the same problem, for any three points on the circle can be used for the construction.

2. *Given three points on a circle, to draw the tangent at any one of them without finding the centre.*



If the tangent is to be drawn at B, any one of the three given points A, B, C, then, with A for centre and any radius, draw the arc DE, cut off between AB, AC. With B for centre and the same radius draw the arc FG equal to DE, starting from BC, and *away from* AB. Join BG.

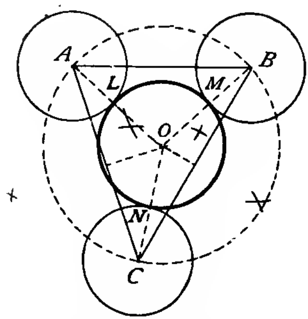
Then BG is the tangent.

PROOF. $\angle GBC = \angle BAC$;
$\therefore$ BG is a tangent. (Euc. III. 32.)

COR. *Draw the tangent at any point of a circular arc without finding the centre.*

Take the given point and any two other points on the arc and proceed as above.

3. *To draw a circle to touch three given equal circles externally.*



Let A, B, C be the centres of the three equal circles.

Find O, the centre of the circle circumscribing the triangle ABC.

Join OA, OB, OC, cutting the three circles in L, M, N respectively.

O is the centre of the circle touching the three circles, and its radius is OL.

PROOF.
$$OA = OB = OC$$
$$AL = BM = CN$$
$$\therefore\ OL = OM = ON$$

COR. *To draw a circle touching three given equal circles internally.*

The construction is the same. The radius is OL', where L' is the point where OA produced cuts its circle.

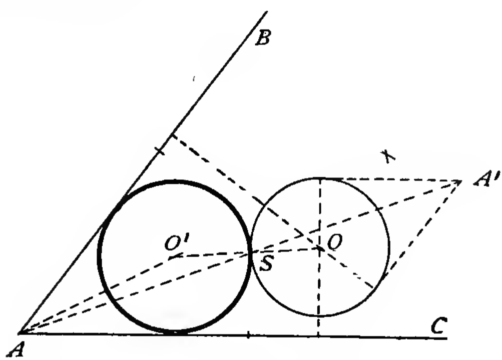
Note.—There are six other circles, three touching two of the equal circles internally and one externally, three touching one internally and two externally. The construction is too complicated for practical purposes.

Miscellaneous Problems on Circles.

It would be easy to enunciate a large number of other problems on the construction of circles satisfying certain conditions; for instance, "*To draw a circle to touch three given unequal circles*". Such problems generally involve very complicated constructions and depend on elaborate geometrical principles. Simple problems will require only the application of the *Six Principles* laid down at the beginning of Chapter IX. (First Part).

The following is a useful case of the more complicated constructions. It depends for its proof on the properties of Centres of Similitude. (*See* page 34.) It must be drawn with great care if it is to be accurate.

4. *To draw a circle to touch two given straight lines and a given circle.*



AB, *AC* are the given straight lines. *O* is the centre of the given circle. Draw tangents to the circle parallel to *AB*, *AC*, as in the figure. Join *A′*, the intersection of

these tangents, to A. Then S, where AA' cuts the given circle, is the point of contact of the required circle.

Join OS (*see* Principle v., page 81, First Part), and let it meet the bisector of the angle BAC in O'.

Then O' is the centre of the required circle, and $O'S$ the radius.

Proof. "The line joining the points of intersection of two pairs of parallel tangents to two circles passes through a Centre of Similitude." Also the point of contact of two circles which touch is a Centre of Similitude. Therefore S is a Centre of Similitude and a point of contact.

Note.—It is obvious from the proof that the other point in which AA' cuts the given circle will be the point of contact of the other circle which touches the circle O *externally*, as well as the two given lines. Its centre can be found just as before. Also another pair of tangents parallel to AB, AC may be drawn, meeting at A'', such that AA'' cuts the circle in two points. These are the points of contact of the two circles which touch AB, AC, and the given circle *internally*.

Thus, in general, four circles can be drawn to touch two given lines and a given circle. The two circles with internal contact will be impossible if the given circle cuts either of the lines AB, AC.

The construction of Problem 4 can be applied in various ways. For instance :—

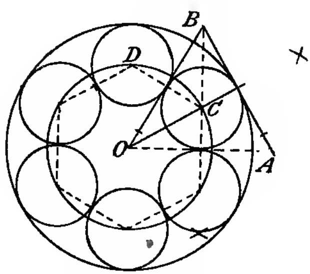
5. *To inscribe in an isosceles triangle three circles, each touching two sides and each other.*

Draw a perpendicular from the vertex to the base, and in each of the triangles so formed inscribe a circle. Then apply Prob. 4 to inscribe a circle in the space above.

Equal Circles Inscribed in or Circumscribed about a given Circle.

The construction for inscribing a circle in a sector of a circle (page 95, First Part) can be extended to the inscribing of a given number of equal circles in a circle ; and an analogous construction can be used for the circumscribing of equal circles. The following examples will show the method to be applied :—

6. *In a circle of 2″ radius to inscribe six equal tangential circles.*

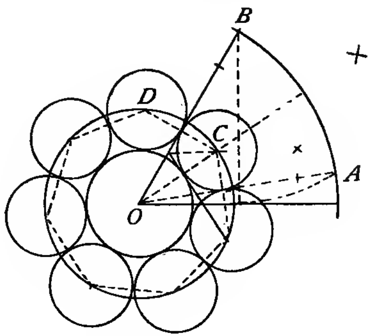


Find two points A, B of the regular hexagon inscribed in the circle. Find the centre C of the circle inscribed in the sector AOB, as in the figure. Draw, with centre O, the circle whose radius is OC. Through C draw a perpendicular to OB to meet this circle at D, and step out CD round the circumference of the inner circle. Thus we get the angular points of another regular

hexagon inscribed in the inner circle. These are the centres of the six circles required.

Join them by dotted lines, and draw the circles.

7. *Draw seven equal tangential circles round the circumference of a circle whose radius is $\frac{1}{2}''$.*



With the centre O of the given circle draw an arc of *large* radius, and on it find A, B, two points of the inscribed regular heptagon.

Join OA, OB, and find C, the centre of the circle which will touch OA, OB and the given circle *externally*.

Draw the circle whose radius is OC. Draw CD perpendicular to OB to meet this circle at D. Step out CD round its circumference, thus getting the centres of the seven circles required.

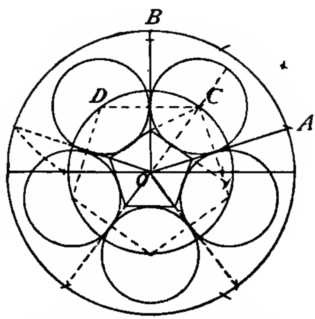
Note.—The object of finding OA, OB with a *large* circle is in order to avoid errors of construction. For then we are working from a larger to a smaller figure, and any errors are diminished.

Equal Circles Inscribed in and Circumscribed about Regular Figures.

Equal circles may be inscribed in or circumscribed about any regular figure, so that the circles touch one another, while each touches one or two of the sides of the figure.

The method is practically the same as the similar problems with respect to a circle.

8. *To circumscribe five equal tangential circles about a regular pentagon of* $\frac{3}{4}''$ *side.*



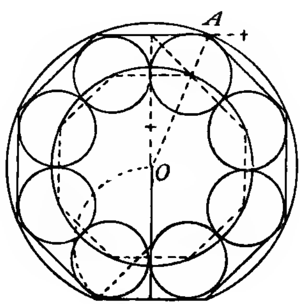
Find the angular points of a regular pentagon inscribed in a circle of *large* radius, and let OA, OB be the radii through two adjacent points.

Draw a regular pentagon of $\frac{3}{4}''$ side concentric with the other (*see* page 29), and find C, the centre of the escribed circle of the triangle formed by OA, OB and one of the sides of the pentagon so constructed.

Draw the circle of which OC is the radius. Draw CD perpendicular to OB to meet this circle at D, and step out CD round the circumference, giving the centres of the five circles required.

Note.—The first centre, C, must be determined with great care. Any inaccuracy can be checked on stepping out CD round the circumference.

9. *To inscribe in a regular octagon of 1″ side eight equal tangential circles, each touching two sides of the octagon.*

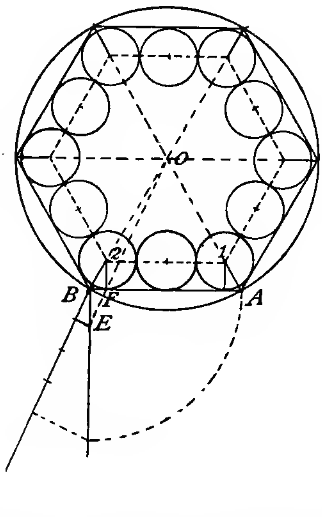


The centre of one of the circles will obviously be on OA, drawn from the centre to one of the angular points, and this circle will touch the perpendicular from O to one of the sides through A. Bisect the right angle so found.

The bisector meets OA at the centre of one of the circles, and the others are found as before.

10. *To inscribe equal tangential circles in a regular figure, so that there may be a given number of circles touching each side of the figure.*

In the example given there are to be three circles touching each side of a regular hexagon.



A consideration of the figure will show that the centres of the three circles, which touch *AB*, lie on the longer side of the rectangle 12*F*, whose sides are in the ratio of 4 : 1, inscribed in the triangle formed by joining *A*, *B* to *O*, the centre of the circumscribing circle.

Draw *BE*, equal to one-fourth of *AB*, perpendicular to *AB*. Join *EO*, cutting *AB* at *F*. (*See* Chap. IV., First Part.) Complete the rectangle *F*21. Then 12 is a line

of centres. Draw the other lines of centres as in the figure. Then the circles can be drawn with a radius $F2$.

Note.—It can be easily shown that if there are n circles touching each side then the ratio of 12 to $2F$ for all regular figures is $2n$ 2 : 1. Thus we have the following ratios for the sides of the rectangle $12F$, inscribed in OAB, for any regular figure:—

The ratio of the sides for 3 circles is 4 : 1 ;
for 4 circles, 6 : 1 ;
for 5 circles, 8 : 1 ;
for 6 circles, 10 : 1, and so on.

Change of Curve.

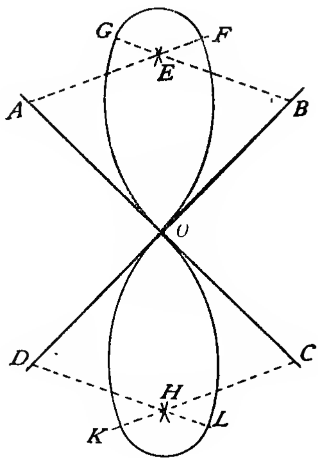
It frequently happens, in geometrical patterns, that a continuous curve consists of arcs of circles with different radii and centres. In such cases the construction for the arcs, which touch one another at the points where the change of curvature occurs, is exactly the same as for the construction of whole circles touching one another. (*See* Chap. IX., p. 86, etc., First Part.) In every case, however, *the line joining the centres of two such arcs must be drawn*, for it passes through the point of contact, that is, the point at which the change of curve occurs, and this point cannot otherwise be accurately determined. When this line is drawn, then the arcs required will be drawn, in each case, *up to the line joining the centres*, and, with ordinary care, there will be no difficulty in obtaining the required curve.

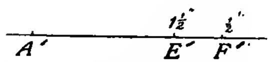
Similar remarks apply to the case where the change is from a circular arc to a straight line. Then, the perpendicular from the centre on to the line must be drawn,

and the arc and the straight line will terminate at the same point on this perpendicular. It is advisable to draw the arc before the straight line.

These points are shown in the following example:—

11. *Draw an 8-shaped figure of circular arcs, such that the larger curves, which have a radius of* 2″, *cut at right angles, while the radius of the curve at either end is* ½″.





Draw two lines *AC*, *BD* cutting at *O* at right angles. Mark off *AO*, *BO*, *CO*, *DO*, each 2″ long. Then *A*, *B*, *C*, *D* will be the centres of the larger arcs.

With these points for centres describe arcs of $1\frac{1}{2}''$ radius, *i.e.*, $2'' - \frac{1}{2}''$, cutting at E, H. Join AE, BE, and produce them, as also CH, DH.

The line AC is the line joining the centres of the two arcs which meet at O and form the curve FOK.

Also, the line AE is the line joining the centres of the two arcs included in the portion OFG. They must meet at F, in AE produced.

Now draw all the large arcs, stopping at F, G, K, L, and then with E, H for centres the smaller arcs FG, KL.

Note 1.—It will be found to increase the accuracy of the construction if the radii used are first marked on a separate line, as $A'E'F'$, and if the distances required are taken off by the compasses from this line.

Note 2.—If the larger curves intersect at an angle other than 90°, the construction is the same, but it must be noticed that AOD *is the given angle of intersection, and not* AOB, and that the arcs intersecting at O will not touch AC, BD.

As an example, repeat the figure when the larger arcs intersect at an angle of 120°.

EXERCISES—IX.

1. Draw with a radius of $1\frac{1}{2}''$ an arc of a circle of less than half the whole circumference, and construct two points, on the part not drawn, without using the centre.

2. With a radius of $1\frac{1}{2}''$ draw an arc of a circle, and draw the tangent at any point of it without finding the centre.

3. Draw an equilateral triangle ABC of $2\frac{1}{2}''$ side. Find two points, E, F, on the circumscribing circle, such that $AE = BF$, without using the centre; and construct the tangents at E, F. Then test your construction by showing that the perpendiculars at E, F to these tangents intersect at the centre of the circle.
4. Inscribe five equal tangential circles in a circle of $2''$ radius.
5. Inscribe three equal tangential circles in a circle of $2''$ radius.
6. About a circle of $\frac{1}{2}''$ radius circumscribe four equal tangential circles.
7. About a circle of $\frac{3}{4}''$ radius circumscribe ten equal tangential circles.
8. In a regular heptagon of $1\frac{1}{2}''$ side inscribe seven equal tangential circles, each touching one side of the heptagon.
9. In an equilateral triangle of $3\frac{1}{2}''$ side inscribe three equal circles, each touching the other two and one side of the triangle.
10. In a regular nonagon of $1''$ side inscribe nine equal tangential circles, each touching two sides of the nonagon.
11. In a regular pentagon inscribed in a circle of $2''$ radius inscribe ten equal tangential circles so that three touch each side.
12. Inscribe in an equilateral triangle of $3''$ side nine equal tangential circles so that four touch each side.
13. Draw three equal circles of $\frac{1}{2}''$ radius with their centres at the vertices of a triangle whose sides are $2\frac{1}{2}$, 2, and $1\frac{3}{4}$ inches respectively. Construct two circles, each touching the three given circles.

14. Copy the figure of eight on page 86, but use radii of 2″ and ·4″.

15. Construct a figure of the same nature as the previous one, but with radii of 2″ and ·9″, and with an angle of 120° between the larger arcs.

16. Draw a square *ABCD* of $1\frac{1}{2}''$ side, and equilateral triangles *ABE*, *CDF*, with vertices *E*, *F* outside the square. With *E*, *F* for centres draw arcs inside the square to terminate at *A*, *B* and *C*, *D*. Construct arcs to pass through these four points, and to be continuous with the arcs already drawn.

17. Draw two lines *AB*, *AC* making an angle of 45° with each other. Mark off $AB = 2\frac{1}{2}''$. Within the angle *BAC* draw a circle of $\frac{3}{4}''$ radius to touch *AB* at *B*. Construct a circle to touch this circle and *AB*, *AC*.

18. In an isosceles triangle, whose equal sides are each 4″ and base 3″ long, inscribe three circles each to touch two sides, the circles which touch the base having equal radii.

19. Draw a circle of $\frac{1}{2}''$ radius. Through *O*, its centre, draw a straight line *OA* $2\frac{1}{2}''$ long. Draw *AB* so that $\angle OAB = 45°$. Construct a circle to touch the given circle and *OA*, *OB*.

20. Draw an equilateral triangle of 3″ side, and in it inscribe a circle of $\frac{1}{2}''$ radius touching two of the sides. Inscribe a second circle to touch two sides of the triangle and the first circle.

21. Inscribe the regular pentagon *ABCDE* in a circle of 2″ radius. With the middle point *O* of the diagonal *AC* for centre draw a circle of ½″ radius. Construct a second circle to touch this circle *internally* and the sides *AB*, *AE*.

CHAPTER X.

AREAS AND FRACTIONS OF AREAS.

TRIANGLES OF EQUAL AREA.

TRIANGLES of the same area but of different shapes can generally be made to depend on one or other of the following propositions :—

(1) Triangles on the same base and between the same parallels are equal. (Euc. I. 37.)

(2) To construct a triangle similar to one and equal to another triangle.

This is possible by Euc. VI. 19 and 20, combined with Euc. VI. 1. The general principle is as follows :—

If a and c are two known lines, the mean proportional, b, can be determined between them, so that

$$a : b :: b : c.$$

Hence $a : c :: \triangle$ on a : similar $\triangle$ on b.

(Euc. VI. 20, Cor. 2.)

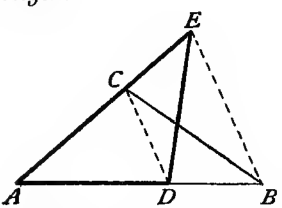
Also $a : c :: \triangle$ on a : $\triangle$ on c, of the same altitude.

(Euc. VI. 1.)

$$\therefore \triangle \text{ on } b = \triangle \text{ on } c.$$

(1) Triangles on the same base and between the same Parallels are equal.

1. *To construct a triangle equal in area to a given triangle (or of given area) and with its base a given length.*



Let ABC be the given triangle. Measure the length of the given base, AD, from A along AB. Join CD. Draw BE parallel to CD to meet AC at E. Join DE.

Then the triangle ADE is equal to the triangle ABC.

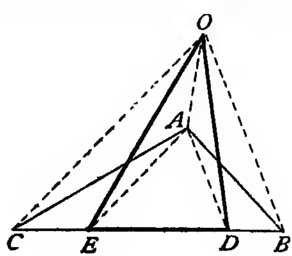
PROOF. $\triangle\, CDE = \triangle\, CDB$. (Euc. I. 37.)

Add to each $\triangle\, ACD$;

$\therefore\ \triangle\, ADE = \triangle\, ABC$.

Note.—This problem is, really, only a particular case of Problem 6, Chapter VII. (First Part)—"*To reduce a given triangle to another of equal area, having its vertex at a given point in one side of the given triangle and its base in the opposite side*". (*See also* Chapter X., Problem 2 (First Part), which can be solved by means of this construction.)

2. *To construct a triangle equal in area to a given triangle, having its vertex at any given point.*



O is the given point either within or outside the given triangle ABC, whose base is BC.

Join OA. Draw AD parallel to OB and AE parallel to OC, to meet BC at D, E. Join OD, OE.

Then the triangle ODE = the triangle ABC.

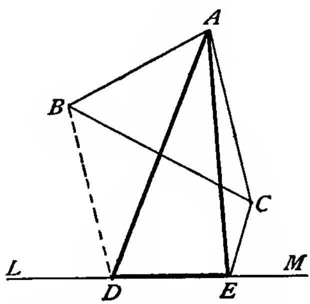
Proof. $\triangle ADO = \triangle ADB$ (Euc. I. 37.)

and $\triangle AEO = \triangle AEC$

$\therefore \triangle DEO = \triangle ABC$.

Note.—This problem is a particular case of Prob. 2, p. 64.

3. *To construct a triangle equal in area to a given triangle, having its base in any given line.*



ABC is the given triangle, LM is the given straight line.

Draw BD parallel to AC to meet LM at D. Join AD. Draw CE parallel to AD. Join AE.

Then the triangle ADE = the triangle ABC.

Proof. $\triangle ADE = \triangle ADC$ (Euc. I. 37.)

$= \triangle ABC$. (Same proposition.)

The three previous constructions can be applied to two classes of problems.

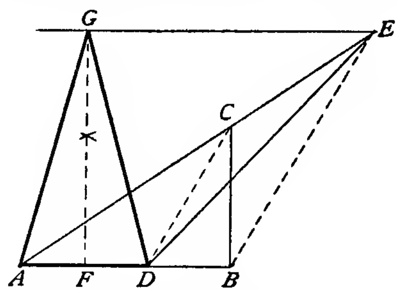
(1) The construction of certain triangles of a given area, satisfying *two* other conditions.

(2) The division of a given rectilinear area into equal parts by lines drawn through a given point.

(*See* Chapter x., First Part, Problems 2, 4, 6, 9.)

The following examples illustrate the methods to be adopted in either case:—

Example 1. *Construct an isosceles triangle of three square inches area, having its base* $1\frac{3}{4}''$ *long.*



Draw the right-angled triangle *ABC*, having *AB* 3″ and *BC* 2″ long. Then the area of *ABC* is 3 square inches.

On *AB* mark off *AD* $1\frac{3}{4}''$ long. Join *DC*. Draw *BE* parallel to *DC*, to meet *AC* at *E*. Through *E* draw a parallel to *AB*, and draw *FG*, bisecting *AD* at right angles, to meet this parallel at *G*. Join *GA*, *GD*.

Then ADG is the isosceles triangle of 3 square inches area.

PROOF.	$\triangle AGD$ is isosceles	(Euc. I. 4.)
	$\triangle AGD = \triangle AED$	(Euc. I. 37.)
	$= \triangle ABC.$	(Problem 1.)

Note.—Similarly other third conditions may be introduced, subject always to the condition that the vertex of the constructed triangle must lie on the parallel through E to the base AB.

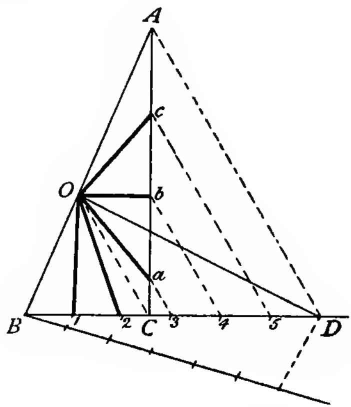
Example 2. *Draw an isosceles triangle ABC, having its vertical angle, A, 120° and its equal sides through A each $2\frac{1}{2}''$ long. Construct a triangle of equal area having its altitude 2″ and its vertical angle 45°.*

Draw the triangle ABC. On AL, the perpendicular to BC, mark off LO, towards A, 2″ long. Then, by Problem 2, construct the triangle DOE of equal area, having DE in BC. On DE construct an arc of a circle containing an angle of 45° (Chapter II., Problem 7, First Part). Through O draw a parallel to BC to meet this arc at G. Then DGE is the triangle required.

Division of an Area into Equal Parts.

The problems given in Chapter X. (First Part) are sufficient for any ordinary subdivision of areas. The two following examples illustrate the construction of Problem 6 (page 104, First Part), which is generally found to be difficult by a beginner:—

4. *To divide a triangle into a given number of equal parts by lines drawn through a given point in one of the sides.*



Let O be the given point in the side AB of the triangle ABC.

Reduce (by Problem 1) the triangle ABC to one of equal area OBD, having its vertex at O and its base in BC.

Divide BD into the given number of equal parts (say 6), and join O to the points which are between B, C (1 and 2 in the figure). Through the rest draw parallels to OC to meet AC (at a, b, c), and join O to the points thus found.

Then the triangle ABC is divided into the required number of equal parts.

PROOF. The same as Chapter X., Problem 6 (First Part).

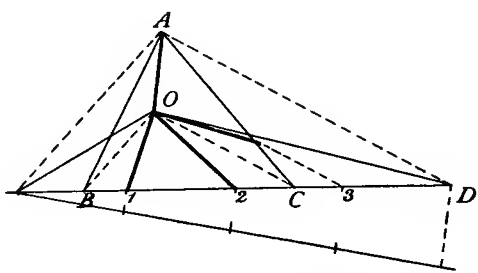
$\triangle\ OB1$ and $\triangle\ O12$ are obviously each $\frac{1}{6}$th of OBD, that is of the given area.

Also $\triangle\ OaC = \triangle\ O3C$, since $a3$ is parallel to OC.

Add to each $\triangle\ OC2$.

$$\therefore \text{ figure } OaC2 = \triangle\ O32$$
$$= \tfrac{1}{6}\text{th of } \triangle\ OBD.$$
$$\text{Similarly } ObC2 = \triangle\ O42$$
$$= \tfrac{2}{6}\text{ths of } \triangle\ OBD;$$
$$\therefore \triangle\ Oab = \tfrac{1}{6}\text{th of } \triangle\ OBD.$$

5. *To divide a triangle into a given number of equal parts by lines drawn through a given point within the triangle.*



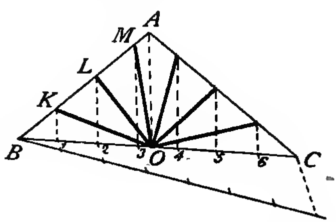
Construct a triangle equal to the given triangle ABC, having its vertex at O, the given point, and its base in BC. (*See* page 92.)

Proceed exactly as in the previous construction, observing that OA is one of the dividing lines.

PROOF. Identical with the previous problem.

Note.—When O is within the triangle, it can be divided up into a given number of parts in an infinite number of ways for, by taking, on BC, any two points whose distance apart is equal to the base of the reduced triangle, and joining the two points so found to O, we get a new triangle equal in area to ABC. If the previous construction be now applied to this triangle, an entirely new set of dividing lines through O will be found.

6. *To divide a triangle ABC into any number of equal parts by means of straight lines drawn through a point O in one of the sides BC.* [Alternative method.]



Divide BC, on which O lies, into the given number of equal parts at 1, 2, 3, etc. Through these points draw parallels to OA to meet AB, AC at K, L, M, etc.

Join OK, OL, OM, etc.

These are the dividing lines.

Proof. $\triangle\, OAL = \triangle\, OA2$ (Euc. I. 37.)

$\triangle\, OAM = \triangle\, OA3$;

$\therefore \triangle\, OLM = \triangle\, A23$

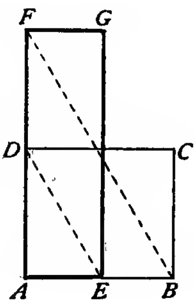
$=$ given fraction of ABC,

and similarly for the other parts of the figure.

Parallelograms of Equal Area.

The principle of Problem 1 is applicable to a variety of cases of parallelograms of equal area.

7. *To draw a parallelogram, given its area and one of the sides.*



Draw any parallelogram *ABCD* of the given area, generally a rectangle. On one of the sides *AB* mark off *AE*, the given length. Join *ED*, and draw *BF* parallel to *ED* to meet *AD* in *F*.

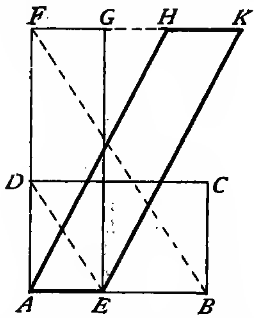
Complete the parallelogram *EAFG*. Then *EAFG* is the parallelogram required.

PROOF. $\triangle\, ABD = \triangle\, AEF$. (*See* Prob. 1, Proof.)

$\therefore$ parallelogram *ABCD* = parallelogram *AEGF*.

This construction has, of course, a third condition available. For instance:—

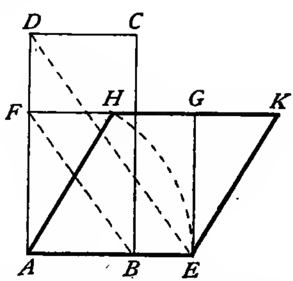
Example 3. *To draw a parallelogram, given the two sides and the area.*



The first part of the construction is the same as above, with AE for one side. Then let an arc be drawn with AH, the second side, for radius, to cut FG produced in H.

Then $AHKE$ is the parallelogram required.

Example 4. *To draw a rhombus whose area and altitude are given.*



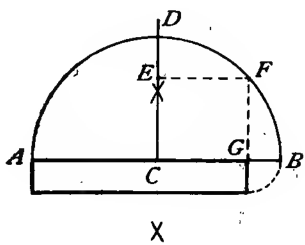
First construct the *rectangle* $AEGF$ of the given area, *having the required altitude* AF *for side.*

Thus, join FB and draw DE parallel to it, to meet AB at E.

Then with A for centre, AE for radius, draw an arc to cut FG at H.

Complete the parallelogram $AHKE$, which is a rhombus of the given area and altitude.

8. *To draw a rectangle of given area and given perimeter.*



Draw AB equal to *half* the perimeter, and on CD, bisecting AB at right angles, mark CE equal to the side of a square of the given area. (*See* Chapter VII., Problem 5, First Part.)

On AB describe a semicircle. Draw EF parallel to AB, to meet this at F. Draw FG parallel to EC.

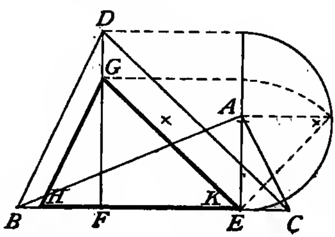
Then AG, GB are two adjacent sides of the rectangle.

PROOF. Euc. II. 14.

(2) The Constructed Triangle similar to one and equal to another Triangle. (Euc. VI. 20, Cor. 2.)

The following is a more general construction for the particular case given in Chapter X., Problem 12 (First Part), and is preferable since it leads to further development.

9. *To construct a triangle equal to one and similar to another given triangle.*



Draw ABC, the triangle of given area, and on one side, BC, draw a triangle BCD of the given shape. Draw the two altitudes AE, DF, and construct the mean proportional FG between them, as in the figure. Draw GH, GK parallel to DB, DC respectively.

Then GHK, which is similar to DBC, is equal to ABC.

Proof. $DF : GF :: GF : AE$; (Construction.)

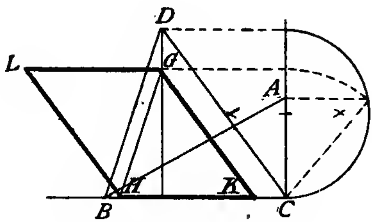
$\therefore\ DF : AE :: \triangle\ DBC : \triangle\ GHK$, (Euc. VI. 20, Cor.)

but $DF : AE :: \triangle\ DBC : \triangle\ ABC$; (Euc. VI. 1.)

$\therefore\ \triangle\ GHK = \triangle\ ABC.$

As extensions of Problem 9 we have the following problems :—

10. *To construct a parallelogram of given area having its sides in a given proportion and the angle between them known.*



Construct a right-angled triangle *ABC* of half the given area. On *BC* draw a triangle *BCD*, having *BC*, *CD* in the given proportion and containing the given angle. Construct, as in the previous problem, the triangle *GHK* equal to *ABC* and similar to *DBC*.

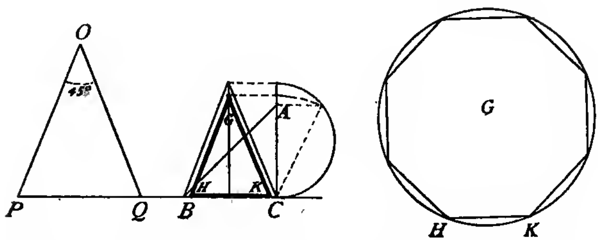
Complete the parallelogram *GKHL*, which is the parallelogram required.

Proof. Identical with the previous proof.

11. *To construct a regular figure of a given area.*

This depends upon the construction of one of the triangles which can be formed by joining the centre of the circumscribing circle to the extremities of one of the sides of the regular figure.

The figure shows the construction for a *Regular Octagon of four square inches area.*



Draw the isosceles triangle OPQ similar to one of the eight triangles formed by joining the centre of the circumscribing circle O to the points of *any* regular octagon. On a portion of PQ produced draw a right-angled triangle ABC equal to one-eighth of the required area, *i.e.*, in this case, having $BC = AC = 1''$.

Construct the triangle GHK equal to ABC and similar to OPQ, as in the figure.

Then GH is the radius of the circle circumscribing the required octagon, and HK is the length of a side of the octagon.

Proof. The same as above.

EXERCISES—X.

1. Draw an equilateral triangle of $2{\cdot}5''$ side, and construct another triangle equal to it in area, having its base $3''$ long and one of its base angles 45°. Measure the longer side.

2. Draw a right-angled triangle of 2 square inches area. Construct another triangle of equal area, having its base 3″ long and its vertical angle 60°. Measure the longer side.

3. In a circle of 3″ diameter inscribe an equilateral triangle. Construct another triangle of equal area having two of its sides 3″ and 2″ long respectively. Measure the included angle.

4. ABC is an equilateral triangle inscribed in a circle of 3″ diameter. O is a point on the circle a quarter of the way between A and B. Construct a triangle of area equal to that of ABC, having its vertex at O and its base in BC produced. Determine the area of each triangle.

5. Draw a square of 2″ side, and construct an isoscéles triangle of equal area with an altitude of $2\frac{1}{2}$″. Measure the base.

6. Draw a rectangle 3″ by $1\frac{1}{4}$″. Construct another rectangle of equal area having one side $2\frac{1}{4}$″. Measure the other side.

7. Draw a parallelogram of 3 square inches area having one side 2″ and one angle 60°. Measure the other side.

8. Draw the rhombus whose sides are each 2″ and area 3 square inches. Measure the shorter diagonal.

9. Draw the triangle in which $AB = 1\frac{1}{2}''$, $BC = 4''$, $\angle BAC = 90°$. Bisect BC at O. Divide the triangle into ten equal parts by lines drawn through O. Measure the length of one of the two shortest dividing lines.

10. Draw an equilateral triangle of 3″ side. Divide it into ten equal parts by lines drawn through the point which bisects the perpendicular drawn from one of the angular points of the triangle to the opposite side. Measure one of the two shortest lines of division.

11. Draw a regular pentagon of 2″ side. Reduce it to a triangle of equal area having its vertex at *O*, the intersection of two diagonals, and its base in one of the sides farthest from *O*. Hence divide the pentagon into five equal parts by lines drawn through *O*. Find the area of the part which is triangular.

12. Draw an isosceles triangle with the equal sides double the base and area equal to that of an equilateral triangle of 2″ side. Measure the base.

13. Construct an equilateral triangle of 4 square inches area. Measure the side.

14. Construct a triangle of 4 square inches area, having its sides in the proportion of 2 : 3 : 4. Measure the shortest side.

15. Construct a rectangle of 6 square inches area with one side double of the other. Measure the shorter side.

16. Construct the rhombus whose angle is 60° and area 3·5 square inches. Measure the side.

17. Construct a regular hexagon of 6 square inches area, and measure the side.

18. Construct a regular pentagon of 6 square inches area, and measure the side.

CHAPTER XI.

GEOMETRICAL PATTERNS.

GEOMETRICAL PATTERNS, consisting of combinations of straight lines and circles, involve a knowledge of the ordinary constructions of geometrical drawing. They may be divided into four classes :—

I. Rectilinear figures; those consisting of straight lines, with or without *intersecting* circles.

In these the Marquois Scales are generally used.

II. Tangential figures; those involving a "change of curve" from one arc of a circle to another touching it, or from a circle to its tangent.

III. Figures in which a "skeleton" figure is first constructed.

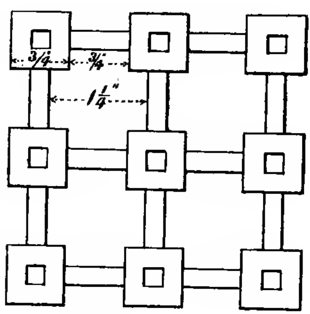
In all these three classes the requisite dimensions are given.

IV. Figures drawn to scale; those in which all the dimensions have to be determined by construction in the figure to be copied.

In every case the pattern must be constructed in pencil first, but, as it has always to be inked in, no dotted pencil lines need be drawn. The instructions given in Chapter XI. (First Part) for inking in and the use of Marquois Scales must be carefully followed. The pencil lines need not as a rule be rubbed out.

The methods to be adopted will be best understood from the accompanying examples. Special attention should be paid to the instructions in II. (p. 110) on "Change of Curve". (*See also* Chapter IX., page 85.)

I. Rectilinear Figures.



All the lines parallel to one direction are to be drawn first, in pencil, by means of the Marquois Scales. These lines should be continuous, and each should be the full breadth of the figure.

Thus it will be seen that twelve parallels must be drawn, the intervals between them being

$$\tfrac{1}{4}'', \tfrac{1}{4}'', \tfrac{1}{4}''; \tfrac{3}{4}''; \tfrac{1}{4}'', \tfrac{1}{4}'', \tfrac{1}{4}'', \tfrac{3}{4}''; \tfrac{1}{4}'', \tfrac{1}{4}'', \tfrac{1}{4}''.$$

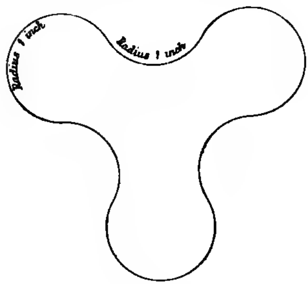
For this purpose use the "40" Marquois Scale, and draw lines at intervals of 10, 10, 10, 30, etc., divisions respectively. When these twelve lines have been drawn, those at right angles to them can be constructed in the same way; but observe that, instead of now drawing the lines across the full breadth of the figure, each line, as it is drawn, may be broken up into parts, so as to indicate the sides of the squares and the bars of the figure.

Thus the first of these lines will consist of three parts, viz., the three sides of the outer squares; the second, $\tfrac{1}{4}''$ away, will consist of five, and so on.

The figure can now be inked in. Care must be taken to avoid inking in those portions of the first set which are not wanted in the final drawing; but after the first two or three are drawn the rest will be found easier. There is less likelihood of mistake if the lines of the second set, which are not continuous, are inked first.

Ink *all* the lines the *same thickness* first. Afterwards ink over again the darker lines, if any, which indicate shade. Notice always that in cases of shadow the light is assumed to come from the top left-hand corner.

II. Tangential Figures.



In this figure it is obvious that the centres of the six arcs of circles to be drawn form a regular hexagon of 2″ side.

Draw this hexagon, and then, with 1″ radius, construct the arcs; but observe that, since a change of curve takes place at the point where two consecutive arcs touch one another, these two arcs *must each start from the line on which they meet*, that is, *the line joining their centres.*

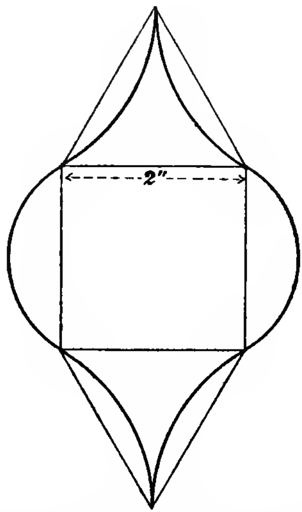
In patterns in which there is a change of curve produced by two or more arcs touching one another, or by an arc and its tangent, the following points must be carefully noticed:—

(1) If the change of curve is produced by two arcs touching one another *the line joining the two centres must in all cases be pencilled in*, and *the two arcs are to be drawn to meet on that line.*

(2) If the change is from an arc of a circle to its tangent, then the *perpendicular from the centre on to*

the tangent must be drawn, and the arc and tangent made to meet on that line.

These points are further illustrated by the second figure.

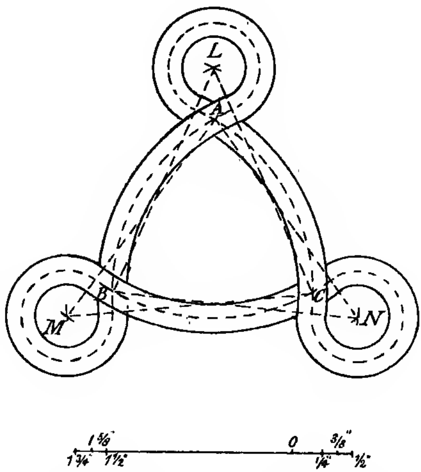


The circular arcs at the top and bottom touch at the vertices of the equilateral triangles. Therefore their centres must lie on the straight lines, through these vertices, parallel to the bases of the triangles. Hence the centre of each is found by drawing these lines and bisecting the two sides of each triangle by perpendiculars.

Again, the centres of the arcs through the corners of the square are the intersections of the lines joining the centres already found to the corners of the square.

III. Skeleton Figures.

In some patterns a "skeleton" figure is required; that is, the curve equidistant from the lines forming the band. Its use will be clear from the following example, in which the dotted arcs form the skeleton. Practically,



of course, they would be *continuous pencil lines and would not be inked in.* The letters are used only to simplify the description.

The dimensions given are that *the radii of the larger circles are* $1\frac{1}{2}''$ *and* $1\frac{3}{4}''$, *and those of the smaller* $\frac{1}{4}''$ *and* $\frac{1}{2}''$.

The centres of the larger circles will be at the middle points of the bands, and therefore $1\frac{5}{8}''$ apart.

Mark $1\frac{5}{8}''$ on a straight line, and at the same time draw, with this radius, the three arcs whose centres are A, B, C.

The distances from these points to the centres of the small circles will be $1\frac{5}{8}''+\frac{3}{8}''$. Mark $\frac{3}{8}''$ on the straight line as shown, and with the radius so found, 2″, and with centres B, C, determine L, and similarly M, N.

Join BL, CL; CM, AM; AN, BN, and with radius $\frac{3}{8}''$ draw three arcs with L, M, N for centres, each terminating, of course, on the "lines of centres" just mentioned.

Now measure $\frac{1}{8}''$ with the dividers, and step it out on the line on either side of the points denoted by $1\frac{5}{8}$, $\frac{3}{8}$. Thus the radii required for drawing the figure are determined with less error than by direct measurement from the protractor, and the whole figure can be drawn, using the six points in the "skeleton" for centres.

IV. Figures Drawn to Scale.

In figures which are to be exactly reproduced the dimensions must be determined by construction. In such cases the magnitude of the dimensions required must be determined by measurements and constructions made on the figure which is to be copied. (*See* Ex. XI., Nos. 31-34.)

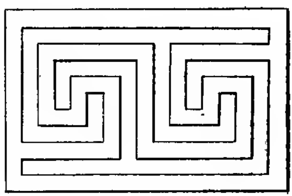
If the figure is symmetrical the axis of symmetry should be drawn first, and the magnitude and position of straight lines measured in relation to this; while the position of the centres of circular arcs and the magnitude of their radii will be found by the construction of Chapter IX., Problem 1 (First Part).

EXERCISES—XI.

Copy the following Geometrical Patterns :—

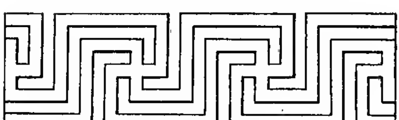
I. Rectilinear Figures.

1.



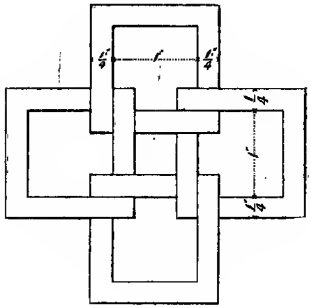
Breadth of band $= \frac{1}{4}''$.

2.



Breadth of band $= \frac{1}{6}''$.

3.



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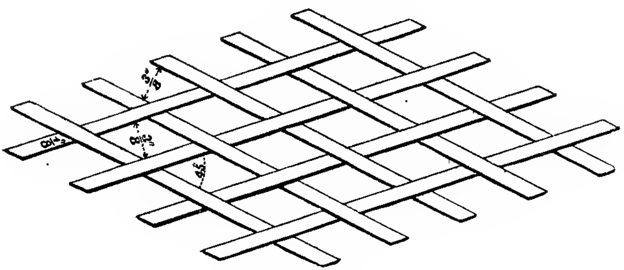

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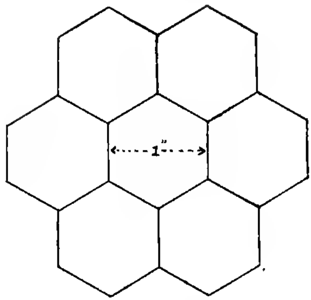

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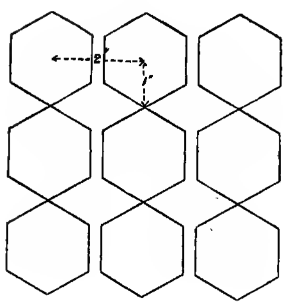

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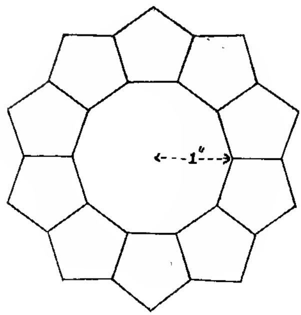

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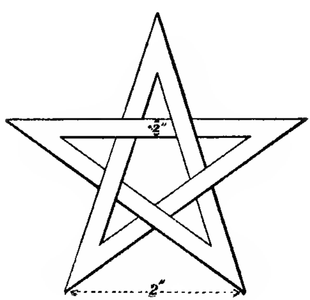

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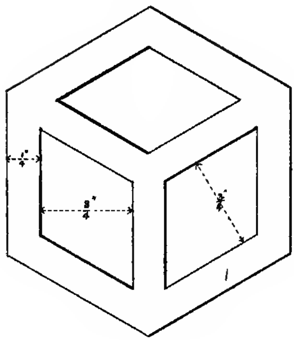
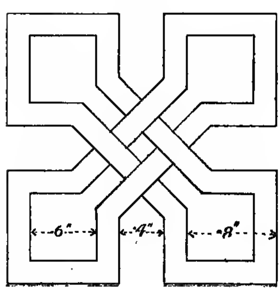
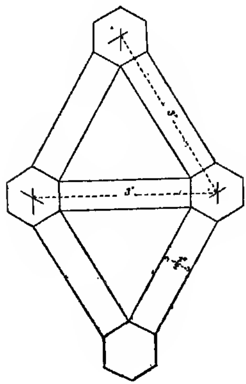


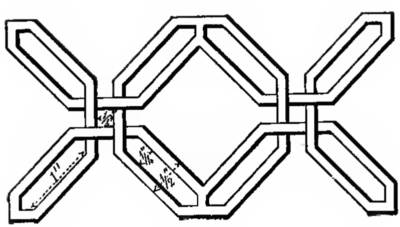
10.



11.

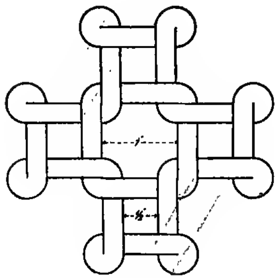


12.



II. Tangential Circles.

13.



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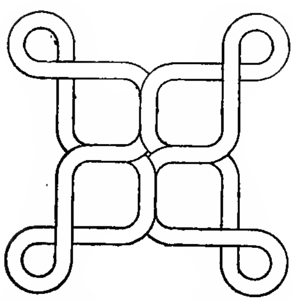
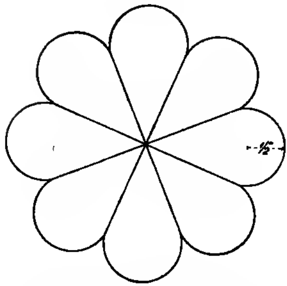


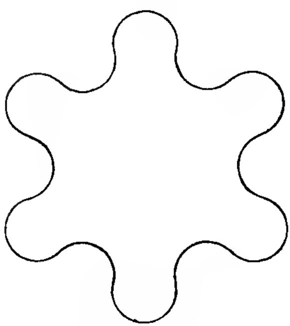
Radii of circular arcs, $\frac{1}{2}''$ and $\frac{1}{4}''$.
Breadth of inner squares, $1''$.

15.



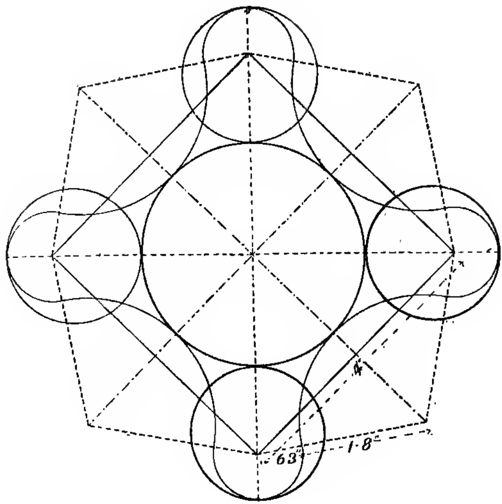
Radius of tangential arcs = ½″.

16.



Radius of circular arcs = ½″.

17.



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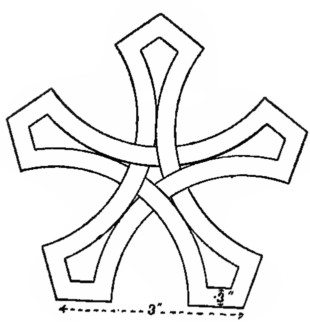

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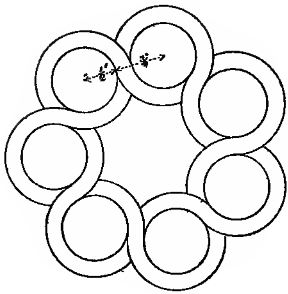

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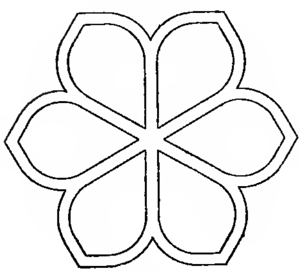
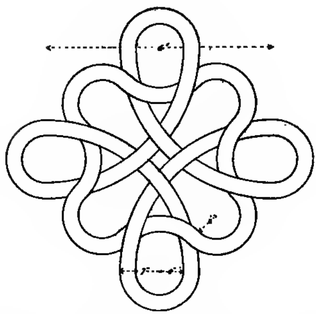


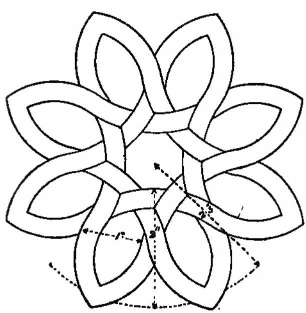
Side of outer hexagon = $2\frac{1}{2}''$.
Breadth of band = $\cdot 2''$.

21.



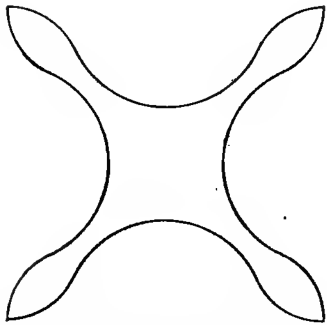
The dimensions given are 4'', ·7'' *and* ·4''.

22.



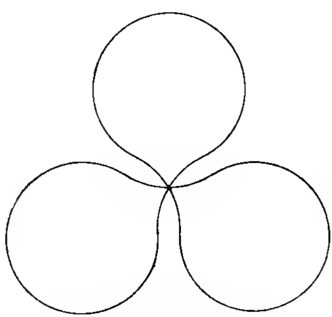
The radii given are 2·5'', 2'' *and* 1''.

23.



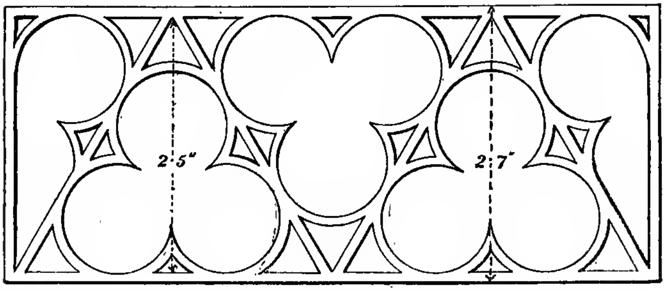
The smaller arcs intersect at right angles.
Radius = ·7″. *Radius of larger arcs* = 1″.

24.



Radius of each arc = 1″.

25.



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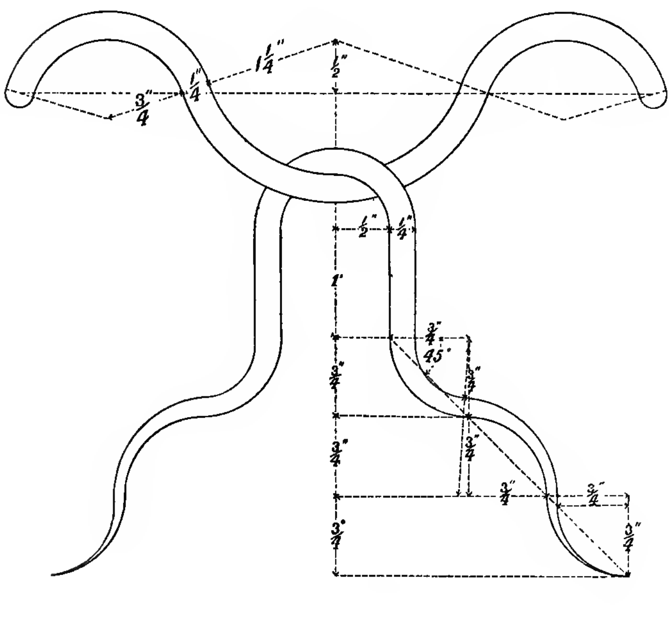
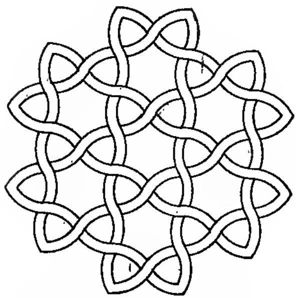


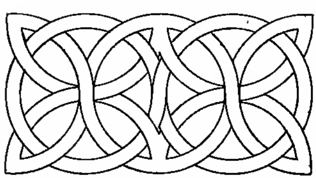
III. Skeleton Figures.

27.



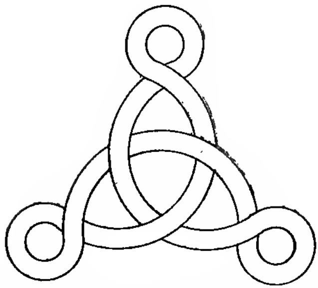
Radius of larger arcs $= \frac{9}{16}''$.
Radius of smaller arcs $= \frac{7}{16}''$.

28.



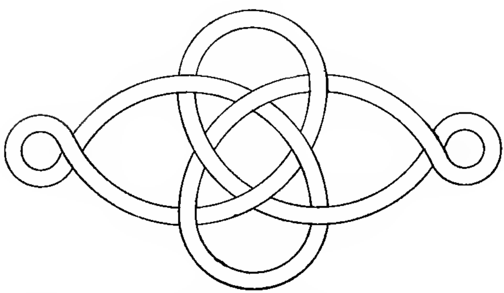
Radii of arcs $= 1''$ *and* $1{\cdot}2''$.

29.



Radii of arcs intersecting round the centre of the figure = ·9″ and 1·1″.
Radii of arcs at the corners = ·4″ and ·6″.

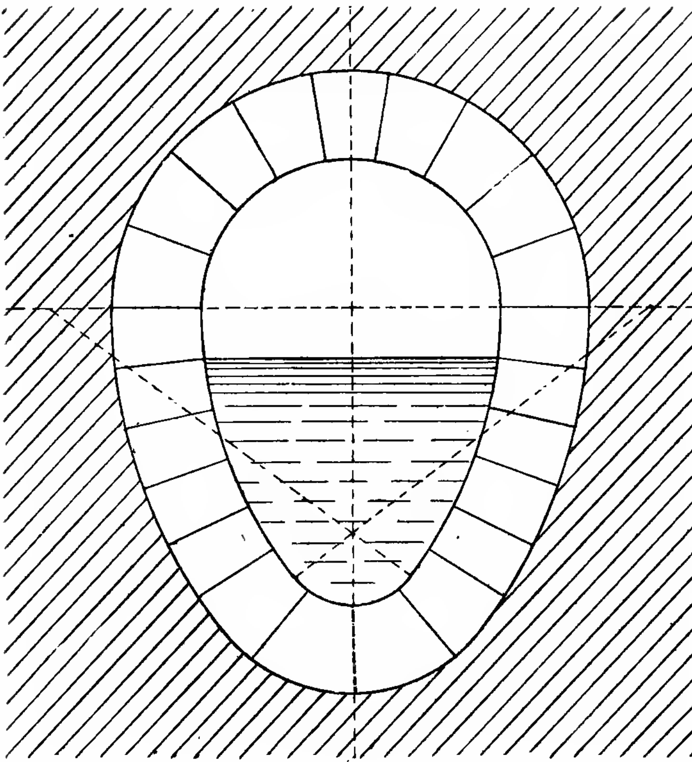
30.



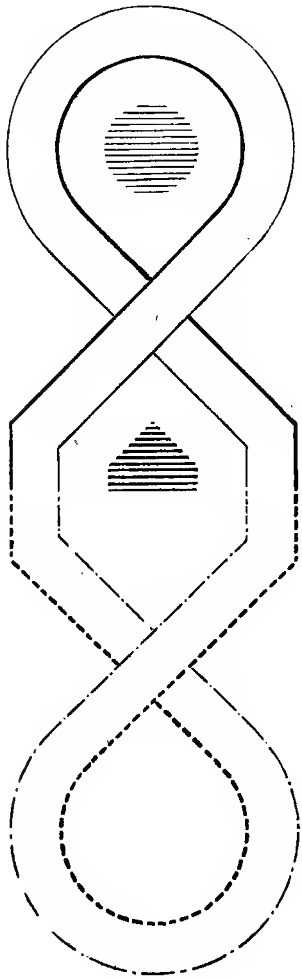
Distances between left and right centres, 4½″; between top and bottom centres, 1½″.
Smallest radii, ·27″ and ·47″. Middle radii, ·65″ and ·85″.
Largest radii, 1·4″ and 1·6″.

IV. Figures to be copied to Scale.

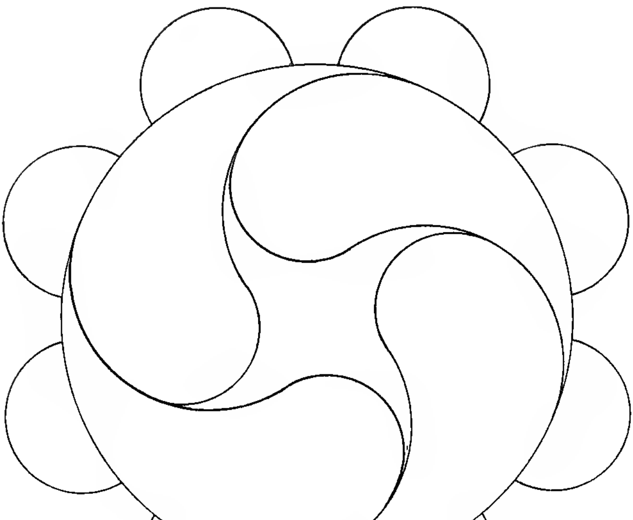
31.



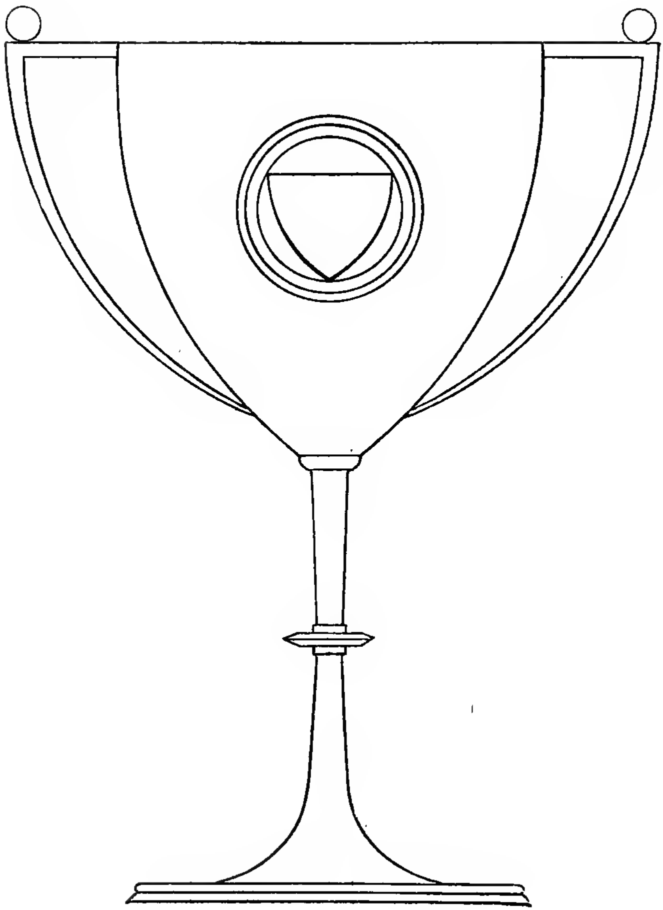
32.



33.



34.

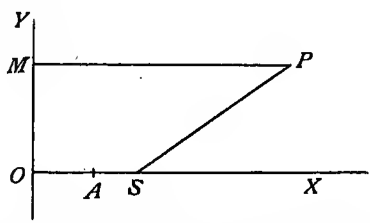


CHAPTER XII.

CONIC SECTIONS—LOCI—ROULETTES AND CYCLOIDAL CURVES.

CURVES, other than the circle, can be, approximately, drawn by constructing a series of points on the curve, and joining these points in order by straight lines.

Of such curves the most common are the Conic Sections —*i e.*, the Parabola, the Ellipse, and the Hyperbola.



DEFINITIONS. If OX, OY are two straight lines at right angles to one another, and if S is a fixed point on OX; then if P be a point such that $\frac{SP}{PM}$ is constant, PM being the perpendicular to OY, the locus of P is a conic section.

The locus of P is

$$\text{a Parabola, if } \frac{SP}{PM} = 1,$$

$$\text{an Ellipse, if } \frac{SP}{PM} < 1, \qquad \text{a Hyperbola, if } \frac{SP}{PM} > 1.$$

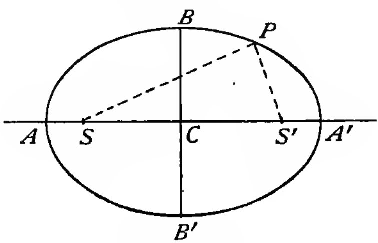
S is called the *Focus*, OY the *Directrix*, $\frac{SP}{PM}$ the *Eccentricity*, and OS produced the *Axis* of the conic section.

If A is the point where the conic section cuts OS, then A is called the *Vertex*.

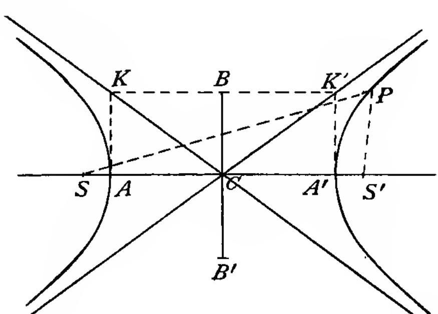
In the Parabola, there is only one focus, and one vertex. In the Ellipse and Hyperbola, there are two foci, S, S', and two vertices, A, A', connected by the following relations :—

In the Ellipse, if P is a point on the curve

$$SP + S'P = AA'.$$



In the Hyperbola, $SP - S'P = AA'$.



If AA' is bisected at C; C is called the centre of the curve, and AA' is called the *Major Axis.*

In the Ellipse, BB', drawn perpendicular to AA' through C to meet the curve, is called the *Minor Axis.*

The Minor Axis of an ellipse can be constructed if the major axis and foci are given, for $SB = AC$.

This follows from the relation

$$SB + S'B = AA'.$$

Therefore, with centre S and radius AC, draw arcs to cut BC, the perpendicular to AA' through its middle point, in B, B'. Then BB' is the Minor Axis.

Similarly, if the axes AA', BB' are given, the two foci can be found by taking B for centre, AC for radius and drawing arcs to cut AA' in S, S'.

In the Hyperbola, a Minor Axis, BB', can be constructed, although the points B, B' do not lie on the curve, for

$$SC = AB.$$

This follows from the relation

$$BC^2 + AC^2 = CS^2.$$

Therefore, if with A for centre and SC for radius, arcs be drawn to cut the perpendicular to AA' through its middle point in B, B', then BB' is the Minor Axis of the hyperbola.

Asymptotes of a Hyperbola. Through B draw a parallel to AA', and let it meet perpendiculars through the vertices A, A' in K, K'. Join CK, CK', and produce them indefinitely in both directions.

Then CK, CK' are called the *Asymptotes* of the hyperbola. The curve gradually approximates to tangency with its asymptotes, but never meets them at a finite distance.

EXERCISES—XII.

1. The major axis of an ellipse is 3″, and the minor axis $1\frac{1}{2}$″. Draw the axes, and find the foci S, S'.
   The measured length of $SS' = 2\frac{1}{2}$″.

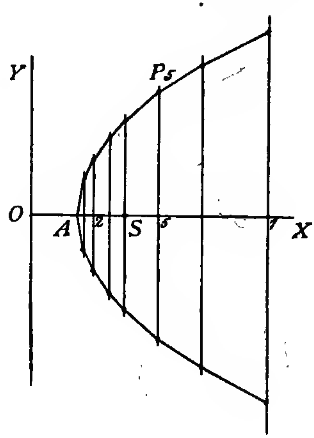
2. The major axis AA' of an ellipse is 4″, and the distance between the foci S, S' is 3″. Construct the minor axis, and show that its measured length is 2·65″.

3. The major and minor axes of a hyperbola are, each, 2″. Construct the asymptotes, and find the foci S, S'. Show that SS' is 2·83″.

4. The angle between the asymptotes of a hyperbola is 60°, and the major axis is 2″. Construct the minor axis, and show that its measured length is 1·15″.

The Construction of a Conic Section.

1. *To draw a Parabola, given the focus and directrix.*



From the focus S draw SO, perpendicular to the directrix OY. (1″ is a convenient length for OS.)

Bisect OS at A. Then A is the vertex of the Parabola, for

$$AO = AS.$$

Now, starting from A towards S, draw a series of perpendiculars to OS, the first close to A, and the others at gradually increasing distances apart. One of these should pass through S, and all must be drawn on both sides of OS. About six will, in general, be sufficient. Let the points where these perpendiculars cut the axis be called 1, 2, 3 . . .

With S for centre, $O1$ for radius, draw arcs to cut the perpendicular through 1 in two points. These are two points on the parabola, and should be clearly marked with a pencil point, or dividers.

Similarly with $O2$, $O3$ . . . for radius, the points on the perpendiculars through 2, 3 . . . can be found.

Join the series of points thus found in order, beginning from A, and the parabola is approximately constructed.

Proof. If P_5 is the point corresponding to the radius $O5$, then, by construction,

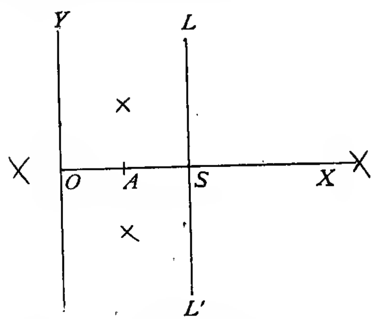
$$\begin{aligned} SP_5 &= O5 \\ &= \text{perpendicular from } P \text{ to the directrix}; \end{aligned}$$

$\therefore$ by definition, P is on the parabola.

Note.—The points 1, 2 must be taken at much smaller intervals than those farther away. The reason for this is, that at A the parabola is perpendicular to the axis. In every class of curve such points must be taken at small intervals for parts of the curve which are perpendicular, or nearly perpendicular, to the line on which these points are marked.

2. *To draw a Parabola whose Latus Rectum is given.*

DEFINITION. The *Latus Rectum* of a Conic Section is the line drawn through the focus perpendicular to the axis, and terminated by the curve.



Draw LL', the given Latus Rectum.

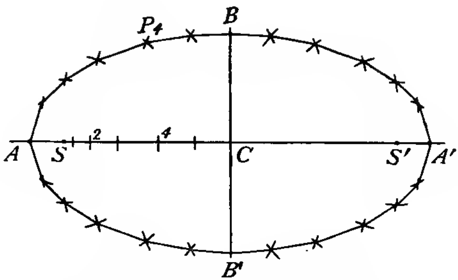
Bisect it by the perpendicular SO. Make SO equal to SL, and draw OY perpendicular to SO.

Then S is the focus, and OY the directrix, and the parabola can be drawn as before.

PROOF. $LS = OS$.

$\therefore$ L is a point on the parabola, of which S is focus, and OY directrix. Also LS is perpendicular to the axis, and is drawn through S, so that LL' is, by definition, the Latus Rectum.

3. *To draw an Ellipse, given the Major Axis and the positions of the foci.*



Let AA' be the major axis, S, S' the two foci, so that AS, $A'S'$ are equal. (Convenient dimensions are $AA' = 3''$, $AS = {\cdot}4''$.)

Bisect AA' by a perpendicular through C, the centre. With S for centre and AC radius draw arcs to cut this perpendicular at B, B'.

Then BB' is the minor axis.

About five or six other points in each quadrant should be constructed. Thus, with centre A, radius up to any point *between* S, C, mark a point on the axis, such as 1. With this same radius, $A1$, draw four arcs of which S, S' are centres. Then with radius $A'1$ draw, with S, S' for centres, four other arcs intersecting the former arcs in four points, which are points on the ellipse, one being in each quadrant.

The series of points thus found—about twenty-two including B, B'—when joined will give approximately an ellipse.

The intervals $S1$, $S2$ must be small for the reason given on page 138, *Note*.

PROOF. If P_4 is a point corresponding to 4, then

$$SP_4 = A4, \text{ by Construction,}$$
$$S'P_4 = A'4$$
$$\therefore SP_4 + S'P_4 = A4 + A'4$$
$$= AA'. \quad (\textit{See} \text{ page } 134.)$$

COR. *To draw an Ellipse, given the major and minor axes.*

Draw the axes AA', BB', bisecting one another at right angles at C.

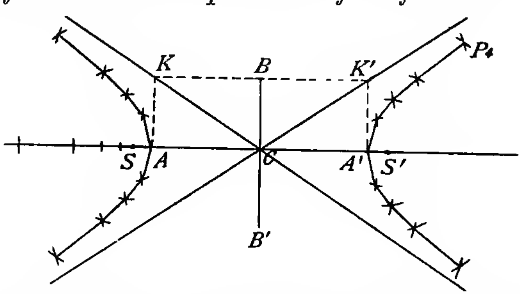
Find the foci S, S' from the relation

$$SB = AC = S'B,$$

and proceed as above.

For other constructions *see* Problem 7, Cor. 2; Problem 9; Problem 13, Cor. 2.

4. *To draw a Hyperbola and its asymptotes, given the major axis and the positions of the foci.*



$AA' = 2''$, $AS = \cdot 2''$ are convenient lengths to assume.

Proceed exactly as in the ellipse, with the following exceptions :—

(1) B, B' are found from the relation

$$BA = SC.$$

(2) The distances $S1$, $12, \ldots$ are to be measured on the axis *away from* C, *instead of between* S and C as in the ellipse.

For the asymptotes, draw KBK' parallel to AA', and let the perpendiculars to it from A, A' meet it at K, K'.

Join CK, CK', and produce them both ways.

Proof. If P_4 be a point corresponding to 4, then

$$\begin{aligned} SP_4 &= A'4, \text{ by Construction,} \\ S'P_4 &= A4 \\ \therefore\ SP_4 - S'P_4 &= A'4 - A4 \\ &= AA'. \end{aligned}$$

(*See* pages 135 and 136.)

Note.—Notice carefully that

(i.) In the Ellipse S is between A and C.
In the Hyperbola A is between S and C.

(ii.) In the Ellipse the points 1, 2, 3, etc., are between S and C.
In the Hyperbola the points 1, 2, 3, etc., are outside CS.

(iii.) In both cases none of the points are taken between A and S.

Cor. *To draw a Hyperbola, given the major and minor axes.*

Draw the axes AA', BB', bisecting one another at right angles at C. Find the foci, S, S', from the relation

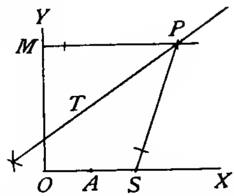
$$SC = AB = S'C,$$

and proceed as above.

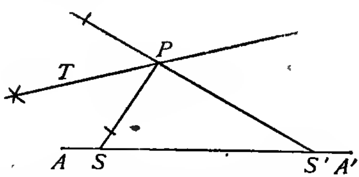
The Tangent to a Conic Section.

5. *To draw the Tangent to a Conic Section at any point P.*

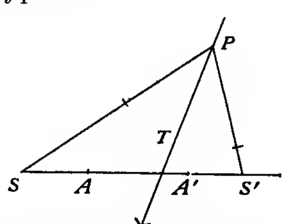
(i.) In the Parabola.



Join SP. Draw PM parallel to the axis OS.
Bisect the angle SPM by PT.
Then PT is the tangent to the Parabola at P.
(ii.) In the Ellipse.



Join SP, $S'P$. Bisect the *exterior* angle between SP, $S'P$.
(iii.) In the Hyperbola.



Join SP, $S'P$. Bisect the *interior* angle between SP, $S'P$.

Proof. Conic Sections.

Cor. *To draw the Normal to a Conic Section at any point P.*

Definition. The *Normal* is the perpendicular to the tangent at the point *P*.

Hence, construct the tangent at the point, and draw the perpendicular through the point of contact.

EXERCISES—XII. (*continued*).

5. Draw a parabola, and at a point *P* construct a tangent. Show, by geometrical drawing, that the perpendicular to *SP*, through *S*, intersects the tangent on the directrix.

6. The above property is true for the ellipse and the hyperbola. Hence, construct a directrix of an ellipse.

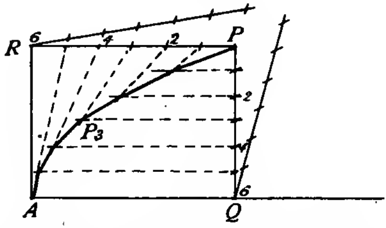
   Also, by measuring *SA*, *AO*, find the eccentricity, $\frac{\text{SA}}{\text{AO}}$, of the ellipse.

7. Draw a hyperbola. Construct a directrix, and find the eccentricity.

8. Prove that tangents to a parabola, at the extremities of a focal chord, intersect at right angles on the directrix.

Conic Sections through given Points.

6. *To draw a Parabola, given the axis, the vertex A and a point P on the curve.*



Draw the rectangle $AQPR$, of which AP is a diagonal, and one side, AQ, is on the axis.

Divide PQ, PR into the same number of equal parts, and, starting from P, number them 1, 2, 3 . . .

If A be joined to any point—say 3—on PR, and if $3P_3$ be drawn parallel to AX, through the corresponding point 3 on PQ, then P_3, the point of intersection, is a point on the parabola.

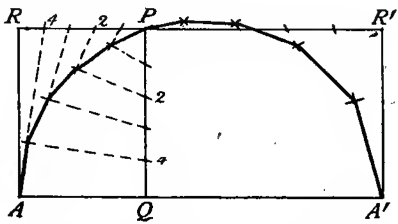
Proof. This can be proved by *Analytical Conics.*

Corollary. *To draw a Parabola, given the ordinate and abscissa of a point on the curve.*

Definition. PQ is the *Ordinate*, and QA the *Abscissa* of the point P.

Note.—Thus—in Dynamics—the path of a projectile can be drawn if the range on a horizontal plane, $2PQ$, and the greatest height, AQ, are given.

7. *To draw an Ellipse, given the major axis AA' and a point P on the curve.*



As in the parabola, draw the rectangle $AQPR$, of which AP is a diagonal, and one side is on the axis AA'.

Divide PQ, PR into the same number of equal parts, and, starting from P, number them 1, 2, 3 . . .

Join A to the points in PR and A' to those in PQ.

Then the intersections of corresponding lines are points on the ellipse.

Points between P, A' can be formed from the rectangle $A'QPR'$ by joining A' to the points of division of PR', and A to those of PQ.

Proof. Analytical Conics.

Cor. 1. *To draw an Ellipse, given the minor axis BB' and a point P on the curve.*

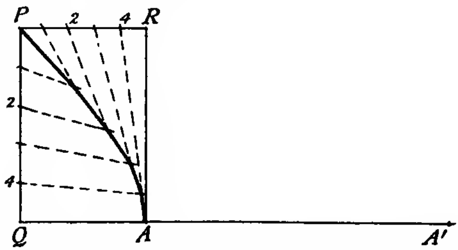
The construction is identical if BB' be substituted for AA'.

Cor. 2. *To draw an Ellipse, given the two axes.* (Alternative construction.)

The constructions given above include the case where the given point is the extremity of either axis.

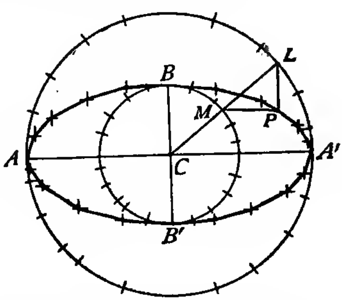
The figure is given in Problem 13, Cor. 2.

8. *To draw a Hyperbola, given the major axis AA' and a point P on the curve.*



The construction is identical with that, for the ellipse, given in the previous problem.

9. *To draw an Ellipse, given the major and minor axes.* (Alternative construction.)



Draw the axes AA', BB', bisecting one another at right angles at C.

With C for centre draw the two circles whose radii are AC, BC.

Points on the ellipse can be constructed thus:—

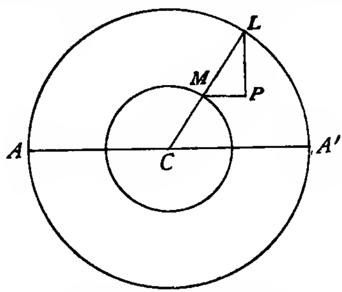
Draw a radius CML, cutting the larger circle at L and the smaller at M.

Draw MP parallel to AC and LP parallel to BC.

Then P is a point in the circle.

PROOF. By the properties of the "Auxiliary Circle".

10. *To draw an Ellipse, given the major axis and a point on the curve.* (Alternative construction.)



With the major axis AA' as diameter describe a circle.

Through P, the given point, draw PM parallel to AA', and PL perpendicular to AA', to meet the circle at L.

Join CL, cutting PM at M.

With C for centre, CM for radius, describe a circle.

Then other points on the ellipse can be constructed as in the previous problem.

EXERCISES—XII. (*continued*).

9. Draw the parabola whose Latus Rectum is 3″.

10. Draw the ellipse whose axes are 4′ and 2″.

11. Repeat this ellipse by the alternative method of Prob. 9.

12. Draw the rectangle $ALPM$, having $AL = 2''$, $LP = 1\frac{1}{2}''$. Draw an ellipse to pass through P, with its vertex at A, and its major axis, 5″, in AL produced.
    Construct the tangent at the point where the line joining M to the centre cuts the curve.

13. Draw the hyperbola whose major axis is equal to the side of a square of 2″ side and whose asymptotes are the diagonals of the square.

14. Draw a parabola to have its vertex at one of the angular points of an equilateral triangle of 3″ side and to pass through the other two points of the triangle.

Conjugate Diameters.

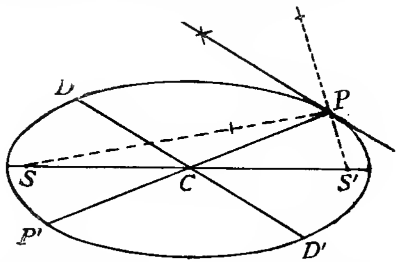
Definitions. Any straight line through the centre of an Ellipse or Hyperbola is a diameter. In a Parabola all diameters are parallel to the axis.

Two conjugate diameters are such that each is parallel to the tangents at the extremities of the other.

Also every diameter bisects all chords parallel to its conjugate.

In the case of the ellipse only are the properties of conjugate diameters suitable to geometrical drawing.

11. *To construct the diameter of an ellipse conjugate to a given diameter, the foci being given.*

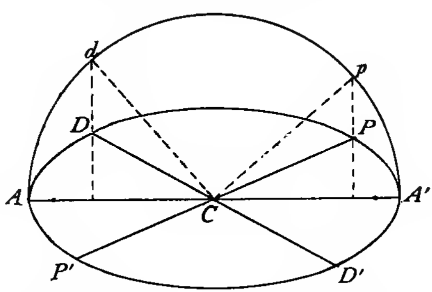


Let the given diameter meet the ellipse at P, P'.

Construct the tangent at P or P' (Problem 5.) Draw DCD', parallel to this tangent, through C, the centre.

Then DD' is conjugate to PP'.

12. *To construct the diameter conjugate to a give diameter of an ellipse, the major axis being given.*



On the major axis, AA', describe a circle.

Through P, the extremity of the given diameter, draw a perpendicular to AA', meeting the circle at p. Join Cp.

Draw Cd perpendicular to Cp, meeting the circle at d. Draw dD perpendicular to AA', meeting the ellipse at D. Join CD.

Then CD is conjugate to CP.

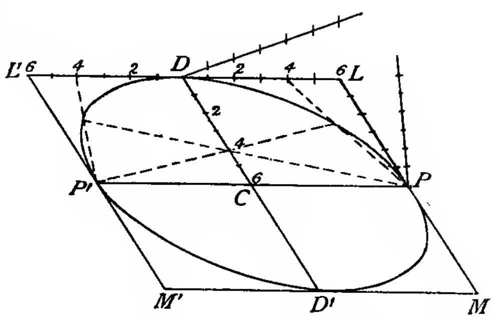
PROOF. Conic Sections. The "Auxiliary Circle".

COR. 1. *To construct the equal conjugate diameters.*

2. *To circumscribe a rhombus about an ellipse.*

In each case construct Cp, Cd, each making an angle of 45° with the major axis AA'; and hence find the points P, D.

13. *To draw an Ellipse, given two conjugate diameters.*



The construction is analogous with that of Problem 7.

PP', DD' are the conjugate diameters intersecting at C, the centre.

Draw the parallelogram $LMM'L'$, such that LM, through P, is parallel to DD', etc.

If DL, DC are divided into the same number of equal

parts, and if P, P' be joined to corresponding points in DL, DC respectively, then the points of intersection are points on the ellipse.

Similarly, points in the other parallelograms of the figure can be constructed.

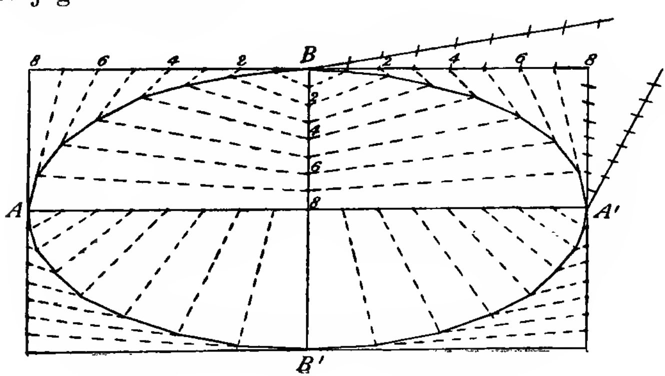
Cor. 1. *To inscribe an Ellipse in a given parallelogram.*

Join P, P' and D, D', the middle points of the sides, and proceed as above.

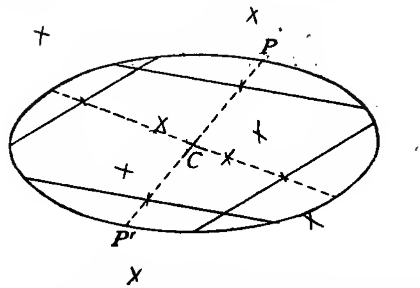
Cor. 2. *To draw an Ellipse, given the major and minor axes.* (Alternative construction.)

The two axes are conjugate diameters, so that the above construction can be applied.

The accompanying figure shows the construction of an ellipse, half by means of the points A, A', the extremities of the major axis, half by means of B, B', those of the minor axis. It is equally applicable to any two conjugate diameters.



14. *Given an ellipse, to find the centre.*



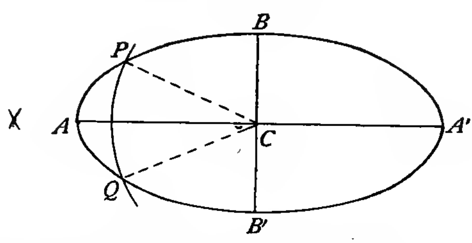
Draw a pair of parallel chords and the chord PP' joining their middle points.

This line is a diameter.

In the same way construct a second diameter.

The intersection of the diameters is the centre C.

15. *Given an ellipse, to construct the axes.*



If necessary find the centre C (Problem 14).

With the centre C of the ellipse for centre and any radius draw an arc cutting the curve at P, Q.

Draw the bisector of the angle PCQ, and the perpendicular to this through C.

These are the axes.

Cor. 1. "*Given an ellipse, to draw a tangent at any point.*"

Find the axes and foci, and proceed as in Problem 5.

Cor. 2. *To draw two tangents to an ellipse which shall be equally inclined to the axis.*

Draw tangents at P, Q in the above figure.

EXERCISES—XII. (*continued*).

15. Two conjugate diameters of an ellipse are 5″ and 3″ and are inclined at an angle of 45° to one another. Construct the ellipse.

    Find the axes of this ellipse, and draw the tangent at an extremity of the latus rectum.

16. The major axis of an ellipse is 4″ long. If the equal conjugate diameters are each 3″ long, construct them and draw the ellipse.

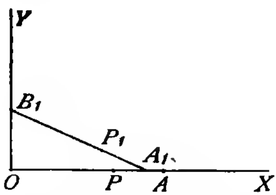
17. Draw a parallelogram having its two adjacent sides 5″ and $2\frac{1}{2}$″ and the included angle 60°. In the parallelogram inscribe an ellipse. Construct the equal conjugate diameters.

18. In a rectangle whose sides are 4″ and 1·75″ inscribe an ellipse. Draw four tangents to the ellipse so as to form a parallelogram.

Loci.

Other curves, besides the Conic Sections, can be drawn by constructing a series of points, which fulfil the given conditions, and by joining these points. The following are two typical cases:—

16. *A straight line, AB, of given length moves so that its extremities always lie on two given lines, OX, OY, at right angles to one another. Draw the locus of a given point P on AB.*



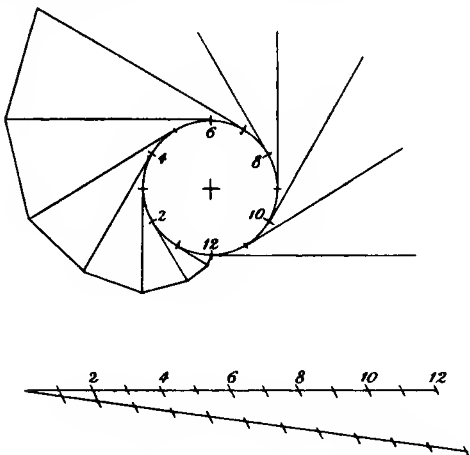
Draw the lines OX, OY. Take two pairs of dividers, and with one measure the distance AB, and with the other the distance PA.

Find the point A on OX, such that $AO = AB$. Mark AP on AO. Then P is a point on the locus.

Draw a series of lines, such as A_1B_1, each equal to AB, and having its extremities on OX, OY, and mark A_1P_1, equal to AP, on A_1B_1. Thus a series of points can be constructed and joined. The intervals, such as AA_1, should be small, near A, and gradually increase towards O.

The locus of P is an *ellipse*. It will be a circle if P is the middle point of AB.

17. *A string wrapped round a circle is unwound in such a way that the part unwound is always a tangent to the circle. Find the locus traced by the extremity of the string.* [The *Involute* of a circle.]



Divide the circumference of the circle into a convenient number of equal arcs, and draw tangents at these points (as in the figure). Twelve is a very convenient number of divisions, for they can all be marked, and the tangents drawn, with the 60 set square, working on a "straight edge".

Calculate the length of the circumference (circumference = diameter × $\frac{22}{7}$). Draw a straight line equal to the length of the circumference, and divide it into the same number of equal parts as on the circle.

Mark off one of these divisions on the first tangent, two on the second, three on the third, and so on.

Join the points so found.

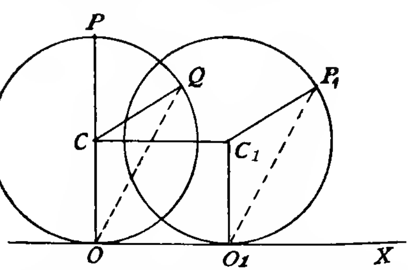
Note 1.—The figure should be repcated, with double the number of divisions.

Note 2.—The tangent at any point of this curve is at right angles to the tangent of the circle on which the point lies.

Roulettes, Cycloids, etc.

If one curve roll along another, then the curve traced by a point connected with the first is called a Roulette.

The simplest case of a Roulette is the curve traced by a point on the circumference of a circle, as the circle rolls along a straight line. This curve is called a Cycloid, and the rolling circle is called the Generating Circle.



Thus, let OX be the fixed straight line, OCP the diameter of the circle, perpendicular to OX, in the initial position; then the curve described by P is a cycloid.

Let C_1 be the position of the centre, when the circle has rolled from O to O_1, and let P_1 be the corresponding position of P.

Then, if, on the first circle, the *arc* PQ be marked off equal to the *line* OO_1, it is obvious that

the arc PQ = the straight line CC_1,

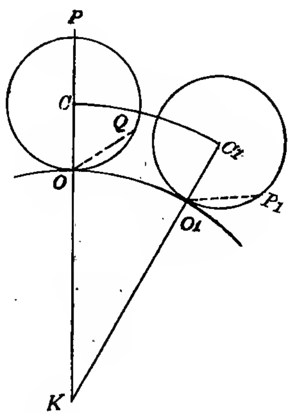
and it can be proved that

CQ, C_1P_1 are equal and parallel,

as also OQ, O_1P_1.

Hence the construction of the cycloid in Problem 18.

The Epicycloid and Hypocycloid are curves of the same class.



Either curve is traced by a point on the circumference of the generating circle, rolling round a fixed circle.

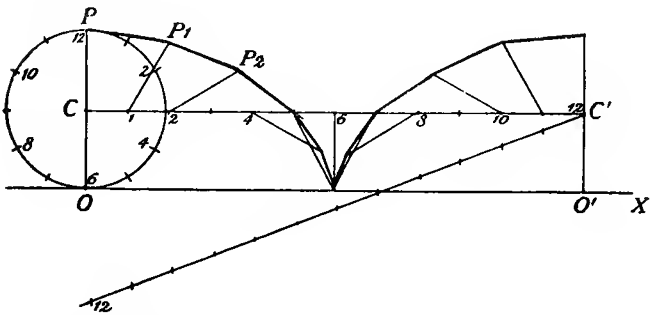
The curve is an epicycloid, if the generating circle is *outside*, and a hypocycloid, if it is *within* the fixed circle.

It may be shown, with the same notation as before, that

arc PQ = arc OO_1,

while OQ, O_1P_1 are equal, *but not parallel.*

18. *To draw a Cycloid.*



OX is the line along which the generating circle rolls. P is the point which traces the cycloid. Initially, the diameter through P is at right angles to OX.

Draw CC' parallel to OX, and mark off CC' equal to the circumference of the circle. (By Arithmetic CC' = diameter $\times \frac{22}{7}$.)

Divide the circumference, starting from P, and CC' into the same number of equal parts. Use set squares. Take 12 divisions, the first time the figure is drawn; afterwards 24. Number the alternate points, as in the figure.

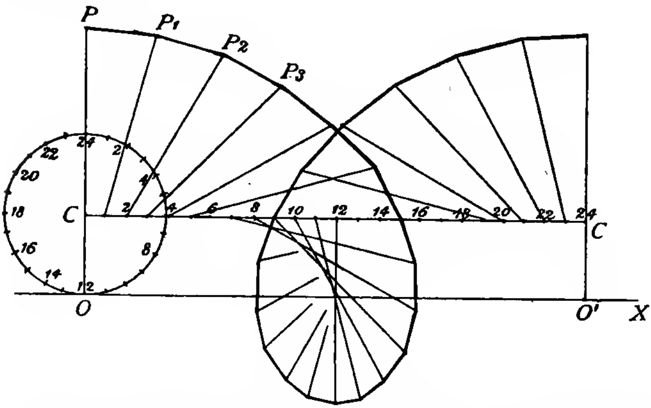
Through 1 on CC' draw $1P_1$ parallel and equal to the radius $C1$.

Through 2 draw $2P_2$ parallel and equal to $C2$, and so on. $P_1, P_2 \ldots$ are points on the cycloid.

Note.—The tangent to the cycloid at any point, such as P_2, is at right angles to the line joining O to the corresponding point 2 on the generating circle in its initial position.

19. *To draw a Trochoid.*

Definition. A Trochoid is the curve traced by a given point inside or outside a circle as the circle rolls along a straight line.



As in the cycloid, draw CC' parallel to OX and equal to the circumference of the generating circle, and divide both * into the same number of equal parts, say 24.

Draw lines through the points 1, 2, 3 . . . on CC' parallel to the radii $C1$, $C2$, $C3$ . . . Mark off on these lines $1P_1$, $2P_2$, $3P_3$ . . . each equal to CP.

Then P_1, P_2, P_3 . . . are points on the trochoid.

Note 1.—In the example given P is *outside* the generating circle and a loop is formed. If P is *inside* the circle, the trochoid does not cut OX, and no loop is formed.

Both cases should be drawn.

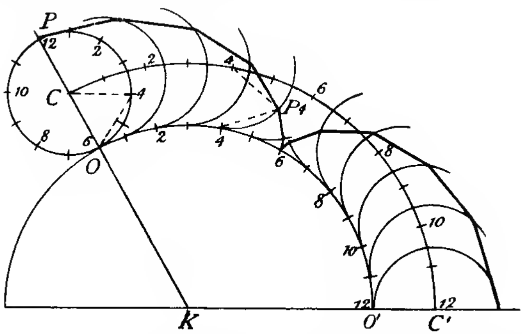
* It will be best to divide the circle of which C is centre and CP is radius, if the generating circle is small, and hence find the points on the latter.

Note 2.—The tangent at any point, such as P_2, of the trochoid is at right angles to the line joining O to the corresponding point 2 on the generating circle.

20. *To draw an Epicycloid.*

Definition. An Epicycloid is the curve traced by a point on a generating circle rolling outside a fixed circle.

In general the diameter of the fixed circle is an exact multiple of the generating circle. Epicycloids are of importance in Mechanics as applied to toothed wheels.



C, K are the centres of the generating and fixed circles in the initial position, with O as point of contact, and P, on CK, the initial position of the point.

Find what fraction the circumference of the generating circle is of that of the fixed circle,* and mark OO', this fraction of the circumference of the larger circle.

Divide both OO' and the whole circumference of the generating circle into the same number of equal parts.

* Or, which is the same thing, that of the radii. In the figure it is $\frac{1}{3}$.

Join K to the points of division of OO', and let the lines so drawn meet the circle whose centre is K and radius KC at a series of points. Number the alternate points on both arcs. With the points on CC' for centres, and radius CO, draw arcs of circles. From any point on the arc OO', such as 4, step out on the circle whose centre is 4 the chord $4P_4$, equal to the corresponding chord $O4$ of generating circle.

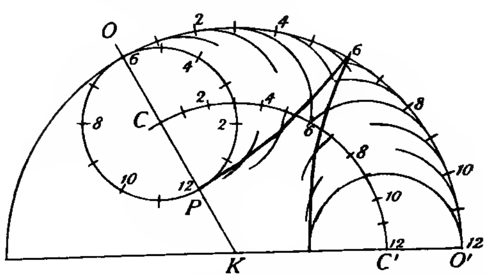
The series of points so found are on the epicycloid.

Note.—The tangent at any point, such as P_4, of the epicycloid is at right angles to $4P_4$, where 4 is the point of contact of the generating circle corresponding to P_4.

21. *To draw a Hypocycloid.*

Definition. A Hypocycloid is the curve traced by a point on a generating circle rolling inside a fixed circle.

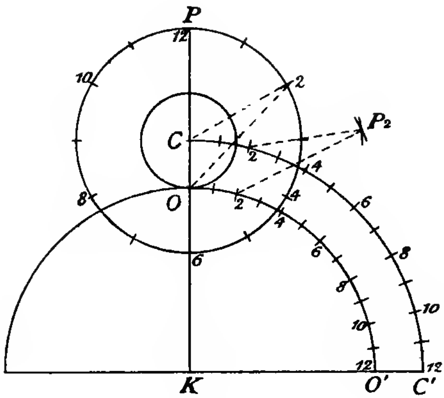
The definition requires the generating circle to be smaller than the fixed circle; if it is larger, the curve described is an epicycloid.

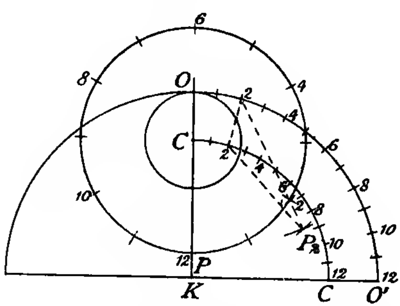


The construction is the same as for the epicycloid, and will be clear from the figure.

22. *To draw an Epitrochoid.*

Definition. An Epitrochoid is the curve traced by a point outside or inside the generating circle as the latter rolls round a fixed circle.





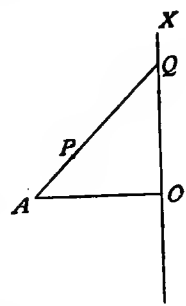
A consideration of the trochoid and the epicycloid together will show the necessary modifications to be made in the construction of the epitrochoid.

The two figures show the construction of the point P_2 for the two cases of the generating circle rolling outside and inside the fixed circles. The ratio of the circumferences in the figures is 1 : 4. Twelve divisions only are marked in the figures, but twenty-four will be required, for the first, for a larger construction.

EXERCISES—XII. (*continued*).

Trace the loci given by the following conditions:—

19. OX is a fixed straight line—A, a fixed point.



AO is perpendicular to OX.
AQ is any straight line through A,

and $PQ = AO$.

Find the locus of P, as Q moves along OX, in both directions from O.

20. OX is a fixed straight line—A, a fixed point.

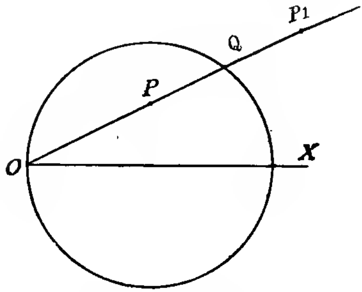
Any line drawn through A cuts OX at Q, and PQ, $P'Q$, equal constant lengths, are marked on AQ and AQ produced.

Find the locus of P and P'.

Note.—Take PQ larger than the perpendicular from A to OX.

This curve is called the *Conchoid of Nicomedes.* OX is an asymptote.

21. OQ is any chord of a fixed circle, through a fixed point O —PQ, P_1Q are equal constant lengths.

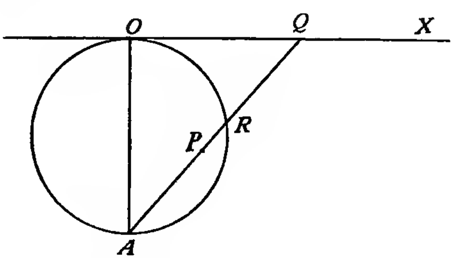


Find the locus of P, P_1, as Q moves completely round the circumference.

This curve is called the *Limaçon.*

22. OX is a tangent to a fixed circle at the extremity of a diameter AO.

If a chord through A to any point Q on OX cut the circle at R, and if $AP = QR$, find the locus of P, as Q moves along OX.



This curve is called the *Cissoid.* OX is an asymptote.

23. Find the locus of the intersection of straight lines which pass through two fixed points on a circle, and intercept on its circumference an arc of constant length.

The locus is a circle through the fixed points.

24. A straight rod AB slides between two rulers fixed at right angles to each other, and from its extremities AP, BP are drawn perpendicular to the rulers. Find the locus of P.

The locus is a circle, whose centre is O.

25. BAC is any triangle described on the fixed base, BC, and having a constant vertical angle. BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle. Find the locus of P.

The locus is a circle through B, C.

26. Find the locus of the centres of the circles which pass through a fixed point O, and touch a given circle.

The locus is a hyperbola or ellipse, according as O is outside or inside the circle.

27. Find the locus of a point which is equidistant from a fixed circle and a fixed straight line.

From A, the centre of the fixed circle, draw AO perpendicular to the fixed line. Bisect BO, the part between the circle and the line, at C. Mark Q, Q' on this line, so that $CQ = CQ'$, and let Q be between C and O. With centre A, and radius AQ, draw an arc to meet the perpendicular to AO, through Q', at the point P. Then P is on the locus.

The locus is a parabola.

28. Given three points on a circle, construct the curve without finding the centre. (*See* Chap. IX. Prob. 1.)

29. Draw an ellipse the major axis of which is 4″, the minor axis $2\frac{1}{2}$″. Draw tangents at the points where the lines bisecting the angles between the axes cut the curve. [Army, June, 1895.]

30. Draw an ellipse with major axis 3″, minor axis 2″, and draw four tangents so as to make a parallelogram containing it. [Army, Nov., 1893.]

31. A reel of cotton (diameter of reel including cotton, $1\frac{1}{2}$″) is fixed in a vice so that it cannot be moved. One turn of the cotton is gradually unwound, the unwound part being kept constantly stretched and at right angles to the axis of the reel. Draw the curve described by the end of the cotton. Draw a tangent to the curve at any point. [Army, Nov., 1895.]

32. A, D are two points 3·7″ apart. AB, DC are two rods each of length 1·8″ revolving about A, D as centres and connected by another rod BC, ·95″ long. Draw the curve traced by the middle point of BC.

The curve is an 8-shaped figure. The intersecting branches are approximately straight lines. [Army, June, 1895.]

33. A railway train is running in a straight line at twelve miles an hour. The door of one of the carriages swings open, so that each point of the door describes with uniform velocity a quarter of a circle about the hinge, and immediately closes again at the same rate, the whole time of opening and closing being half a second.

The width of the door is 27″. Draw the curve traced out by a point on the edge of the door. Scale (which need not be constructed), $\frac{1}{2}''$ = 1 foot. [Army, Nov., 1894.]

A curve of the same class as the cycloid, but half concave, half convex. The distance travelled by the hinge, the centre of the generating circle, is 8·8 *feet between the time of opening and shutting.*

34. A circle whose centre is C and radius $CO = \cdot 4''$ rolls externally round a fixed circle whose centre is K and radius $KO = 1\cdot 6''$. Draw the epitrochoid traced by a point P connected with the first circle and 1″ from its centre as this circle rolls all the way round the fixed circle.

Note.—The curve traced will have four complete branches. Twenty-four points must be constructed in each branch.

35. With the data of the previous problem draw the epitrochoid when the circle whose centre is C rolls round the *inside* of that whose centre is K.

EXAMINATION PAPERS

SET IN ARMY EXAMINATIONS.

I.

TIME ALLOWED—THREE HOURS.

N.B.—*The figures are to be neatly drawn in clear, fine pencil lines. If time allows, they may be inked in with Indian ink. The double accent (") signifies inches. All construction lines must be shown either in pencil or dotted in Indian ink.*

1. The distance between two towns is $27\frac{1}{2}$ miles, and measures on a map 3·25 inches. Construct a diagonal scale of miles and furlongs for the map showing 50 miles. Show all your calculations, figure your scale properly, and write above it its representative fraction.

   By means of this scale draw a line 23 miles 5 furlongs long.

   (1 mile = 8 furlongs = 1,760 yards.)

2. Construct a triangle on a base of 3·5", having its vertical angle 70°, and its altitude 2·25".

3. Draw two straight lines AB, AC of indefinite length, and having the angle BAC 50°. Take a point D in AB, 3" from A, and a point E in AC, 2·25" from A. Describe a circle that shall touch AB in D, and cut AC in E, and a

second point F (to be determined). Show by a separate geometrical construction that $AE \times AF = AD^2$.

4. On a plan 1,250 yards are represented by 15·5 inches. Draw a comparative scale of French metres for the plan, showing 500 metres, and divided to show distances of 10 metres. Show all your calculations, figure your scale properly, and write above it its representative fraction.

(1 metre = 1·0936 yards.)

5. Describe a circle of ·75″ radius, and about it describe seven equal circles, each touching two others and the original circle.

6. A line 4 inches long moves with its extremities in two straight lines at right angles to each other; determine the curve traced by a point in the line 1·25″ from its end.

7. Draw the geometrical pattern shown in the accompanying diagram,* adhering strictly to the figured dimensions.

(*N.B.*—This should be inked in, reproducing the fine and thick lines, as shown in the diagram.) [1883.]

II.

1. Draw a diagonal scale of $\frac{1}{1053}$ to measure single feet. Show 500 feet. Figure your scale properly, and show all your calculations. Show by two small marks on the scale the points you would take in order to measure off a distance of 373 feet.

* The pattern on p. 108 was set.

2. Construct a square of 3″ side, and trisect it by lines drawn parallel to one of its diagonals.
3. With a radius of ·75″ describe two circles having their centres 2·75″ apart, and with the same radius describe a third circle having its centre 3″ from the centre of one of the first two, and 4·5″ from that of the other. Describe a circle touching these three equal circles.
4. A map is drawn to a scale of 6 inches to an English mile. Draw a plain scale of Spanish yards for the map, showing 2,000 Spanish yards, and divided to show distances of 50 Spanish yards. Show all your calculations, figure your scale properly, and write above it its representative fraction.

   (1 Spanish yard = ·9277 English yard.)
5. Construct an equilateral triangle of 4 square inches.
6. Given a circle of 1·5″ radius, take a point *A*, 3″ from its centre, and a point *B*, in the circumference, 2″ from A. Describe a circle which will pass through the point *A*, and touch the given circle at *B*.
7. Draw the geometrical pattern given.* [1883.]

III.

N.B.—*Each figure should be neatly drawn in clear, fine pencil lines on the same page as the question. Questions* 1, 2, *and* 3 *must be inked in with Indian ink.*

The solutions must be strictly geometrical, and all lines of construction should be shown dotted, and unnecessary lines rubbed out.

No writing whatever is allowed on the blotting paper.

* The pattern on page 110 was set.

1. Copy accurately the given figure.
   (This must be inked in, or no marks will be given. If you have not time to finish the whole, ink in a portion.)

*2. An English map is drawn to a scale of 6 inches = 1 mile. Construct a scale for use with this map in French measure. One metre = 39·37 inches (nearly). Give the representative fraction, and indicate on the scale a distance of 1,470 metres.

*3. In a certain country 10 units of area equal an acre. Make a scale for use in this country which will read to tenths and hundredths on the diagonal part.
   The representative fraction is $\frac{1}{528}$.
   Indicate by two marks on the scale 3·78 units of length.

4. AB is a straight line. P, Q are two points on the same side of AB, distant $\frac{3}{4}''$ and $1\frac{1}{2}''$ respectively from it. $PQ = 2''$.
   Describe a circle which shall pass through P and Q and touch AB.

5. $ABCD$ is a quadrilateral. $AB = 2''$, $ABC = 120°$, $BAD = 110°$, $BC = 2''$, $AD = 3''$. Bisect the quadrilateral by a straight line drawn through A.

6. Describe a triangle whose area is equal to that of a square of $2''$ side, its base being $3''$ and its vertical angle $55°$.

[1891.]

* The scales must be accurate, neatly finished, and inked in, construction lines must be shown, and all calculations clearly written near the scale.

IV.

1. Copy the given figure.

   (This must be inked in, or no marks will be given. If you have not time to finish the whole, ink in a portion.)

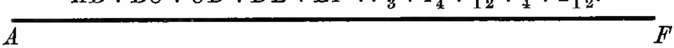
*2. The actual distance between two places marked on a Russian map is known to be 5·2 versts, and you find that these two places measure 4·56 inches apart. Construct a plain scale for the map of versts and sachines showing altogether 8 versts, and least divisions 50 sachines. Calculate and write down above the scale its representative fraction.

   (1 verst = 500 sachines = 1,166·6 English yards.)

3. Given representative fraction $= \frac{1}{9 \cdot 5}$. Construct a *diagonal* scale to measure 5 feet and to show $\frac{1}{8}$ths of an inch. Show by two small dots on the scale a distance of 3 feet $4\frac{7}{8}$ inches.

4. Divide the line AF from A to F in the following proportions:

   $AB : BC : CD : DE : EF :: \frac{2}{3} : 1\frac{1}{4} : \frac{5}{12} : \frac{3}{4} : 2\frac{1}{12}$.



5. Describe a circle of $1\frac{1}{2}$ inches radius; inscribe in it a regular pentagon *abcde* (lettered in order). Join the middle points of *ab* and *cd*, and about the four-sided figure thus obtained circumscribe a circle.

6. Construct a square of 2 inches side, and upon one of its diagonals as base construct an isosceles triangle with area : area of square :: 7 : 5. Measure and write down the length of a side of this triangle in inches and decimals of an inch. [1891.]

* See footnote on page 171.

V.

1. Copy the given figure.

2. The distance between two points is known to be 970 yards, and measures on a military plan 11·64 inches. Draw a scale of yards for the plan showing 500 yards. Show single yards by the diagonal method, give all necessary calculations, figure your scale properly, and write above it the representative fraction. Show by two small marks on the scale the points you would take to measure off a length of 237 yards.

3. A distance of 20 Flemish miles is represented on a Flemish map by $8\frac{1}{2}$ English inches. Draw a scale of English miles for the map, showing single miles up to 60. Assume the Flemish mile = 3·6387 English miles.

4. Draw a scalene triangle ABC, and divide it into three equal parts by lines drawn through a point P within the triangle.

   Test the accuracy of your work by finding the area of the given triangle and the area of one of the parts.

5. Draw two lines AB, CD, not meeting one another, and, without producing them, bisect the angle between AB, CD.

   Test the accuracy of your construction by producing all three lines till they meet.

6. On a base BC of $4\frac{1}{2}''$ describe a triangle ABC, having its sides AB, AC, $3''$ and $3\frac{3}{4}''$ respectively.

   In this triangle inscribe a rectangle with one side double of the other, the longer side being in BC, and the other two points on AB and AC. [1891.]

VI.

1. Describe a square of half an inch side; about this describe a circle; about the circle describe a second square, etc., until six squares, all with their corresponding sides parallel, and six circles are drawn.

2. Draw a plain scale of statute miles, 5 miles to an inch. Give the representative fraction.

   Draw also a comparative scale of geographical miles. Show 20.

   (1 geographical mile = 6,083 feet.)

3. Round a circle of half an inch radius draw seven equal circles, each touching two and the given circle.

4. Take three points A, B, C. It is required to draw the triangle of which these are the feet of the perpendiculars from the angles to the opposite sides.

5. With R.F. = $\frac{1}{88}$ construct a diagonal scale to measure feet and inches. Show 40 feet. Show by two small circles on the scale the points to be taken to measure a distance of 17 feet 8 inches.

6. On a straight line $2\frac{1}{2}$ inches long and on the same side of it construct an equilateral triangle and a regular pentagon. Reduce the space included outside the triangle and inside the pentagon to a triangle of equal area. [1890.]

VII.

1. Copy the given figure.

2. Determine the representative fraction of the scale of a map 12 inches long and 14 inches wide which represents an area of 40 acres. Draw a scale for the map.

3. Define a comparative scale. Draw a scale of Milan miles comparative to a scale of English miles, 8 miles to the inch.

   (1 Milan mile = 1,808·81 English yards.)

4. Construct the equilateral triangle whose altitude is 1·5″. Draw a square equal to it in area. Also draw a square of which the diagonal is 1·5″.

5. Draw two lines which converge but do not meet. Between these lines describe three circles touching one another in succession and also the converging lines.

6. On a base of ·75″ long describe a regular octagon.

[1892.]

VIII.

[*The first question should be attempted first, and will not be marked unless inked in. The Problems may be left in pencil, which should not be too faint ; all constructions being shown (dotted), and unnecessary lines rubbed out.*]

1. Reproduce, as accurately as you can, the given figure.

2. Find, by construction, angles of 30°, 52½°, and 105°.

3. Draw a straight line *AB* 3½″ long. With the extremities *A* and *B* as centres describe two circles whose radii are to each other as 5 to 7 and whose circumferences are half the length of *AB* distant from each other.

   Describe a third circle of 4″ radius to touch and contain both the other circles.

4. The sides of a triangle are respectively 2½ and 3 times as long as the base, and the altitude is 1½″. Construct the triangle.

5. Construct by any method a regular pentagon of $1\frac{1}{2}''$ side, and reduce it to a rectangle of equal area of which the length is five times the breadth.

SCALES.

[*These scales must be accurate, neatly finished, and inked in, and the calculations clearly shown on the same page as the scale.*]

6. Construct a scale of 9″ to a mile, the primary divisions being furlongs and the secondary divisions 5 perches each.

(1 perch = $5\frac{1}{2}$ yards. 1 mile = 8 furlongs = 1,760 yards.)

Construct a comparative scale of paces, primary divisions 200 paces, secondary divisions 10 paces. Show 2,000 paces. (1 pace = 30″.)

7. Upon a Chinese map a known distance of 67 chang 3 chih scales 4·1″. Write down the representative fraction of the map, and construct a diagonal scale for it, showing li, chang, and chih.

(1 li = 180 chang = 1,897 English feet.
1 chang = 10 chih.) [1889.]

IX.

1. Copy as accurately as you can in ink the given design.

SCALES.

[*These scales must be accurate, neatly finished, and inked in, and the calculations shown with the scale.*]

2. Construct a plain scale (not diagonal) of nautical miles, showing cables and least divisions 10 fathoms.

(1 N. mile = 10 cables = 1,000 fathoms = 6,086 feet.
Representative fraction = $\frac{1}{12000}$).

3. Given that 1·45″ represent 203 metres on a French map, construct a scale of kilometres for the map, with a diagonal scale showing metres.

State the representative fraction, and show by two small dots on the scale a length of 1,062 metres.

(1 metre = 1·0936 English yards.
1 kilometre = 1,000 metres).

4. Draw two straight lines cutting each other at an angle of 110°, and describe a circle of $1\frac{1}{2}''$ radius touching them both.

Questions 5 *and* 6 *are alternative. Both must not be attempted.*

5. Two circles whose centres are 3″ apart and their radii $\frac{1}{2}''$ and 2″ respectively touch a straight line upon the same side of it. From the centres of the circles draw two straight lines meeting in a point on the first line and making equal angles with it.

6. Upon a straight line *AB* 2·7″ long construct a square *ABCD*. Upon the opposite side of *AB* construct a triangle *ABE*, of which the side *AE* is 2·3″ and *BE* is 1·6″ long. Upon the side *AD* outside the square construct a rectangle *AFGH*, having the side *AF* in *AD* and = 1·9″ and *AH* = ·75″. Reduce the whole figure *AEBCDFGH* to a triangle, having its base in *AB* or *AB* produced and one end of the base at *B*. Write down the area in square inches of this triangle.

12

Questions 7 and 8 are alternative. Both must not be attempted.

7. A triangle has two sides 2·3″ and 3″ long respectively. Find by construction and write down in figures to one place of decimals what the length of the third side must be in order that the triangle may be equal in area to a square of 1″ side.

8. Construct a trapezium $ABCD$ having the diagonal $BD =$ 2·9″, the sides AB, $AD = 2{\cdot}6''$ each, and the sides CB, $CD = 4{\cdot}1''$ each.
   Inscribe a square in the trapezium. [1890.]

X.

1. Copy the accompanying figure.

2. Draw a diagonal scale of yards to read also feet and inches, the representative fraction being $\frac{1}{53}$. Show 10 yards.
   [*This scale* MUST *be figured and completed in ink. Great importance will be attached to neatness.*]

3. A vertical pole 100 feet high casts a shadow on the ground, the sun's rays being inclined to the horizontal at an angle of 38°.
   If the pole is represented by a line 3 inches long, construct a scale to measure the shadow, and find its length in feet by means of the scale.

4. Construct an isosceles triangle ABC whose base $BC = 2{\cdot}5$ inches and vertical angle 42°.
   Bisect this triangle by a line drawn parallel to the side AC.

5. Inscribe a square in a triangle, the sides of which are respectively $2\frac{1}{2}$, $3\frac{1}{2}$, and 4 inches.
6. Construct a regular pentagon of 2 inches side and a triangle whose area : area of pentagon : : 1 : $\sqrt{2}$.
7. Having given a straight line AB and two points C, D, both on the same side of AB, draw a circle passing through C, D and touching AB.
8. Draw an ellipse with major axis 3 inches, minor axis 2 inches, and draw four tangents so as to make a parallelogram containing it. Extra credit will be given if the tangents are not at right angles or parallel to the axes. [1893.]

XI.

1. Copy the given figure.*
2. Construct a scale of yards to read feet and inches, representative fraction $\frac{1}{11}$. Show by marks upon your scale a length of 1 yard 2 feet 7 inches.

   [*This scale* MUST *be figured and completed in ink. Great importance will be attached to neatness.*]
3. Construct a diagonal scale of miles, furlongs, and chains. Take $1\frac{3}{8}$ inches to represent 8 furlongs. Give the representative fraction. Make the scale to read 5 miles. Show by two marks upon your scale the points you would take to measure off a length of 3 miles 1 furlong 7 chains.
4. Construct a nonagon (a regular nine-sided figure) of $1\frac{1}{2}''$ side, and inscribe a circle in it.

* Ex. XI., No. 34, was set.

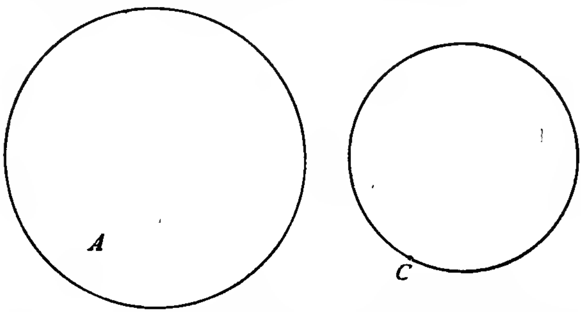
5. Construct a triangle, two of its angles being 40° and 105° respectively, and the radius of its inscribed circle $\frac{3}{4}$ inch.

6. Draw two triangles whose sides are $2\frac{1}{4}''$, $2''$, $1\frac{1}{8}''$ and $3\frac{7}{8}''$, $3\frac{1}{8}''$, $1\frac{3}{4}''$ respectively.

Construct a triangle equal in area to the sum of the areas of these two triangles having its base 3 inches long.

7. Describe two circles touching the smaller circle at the point C, and also touching the circle A.

N.B.—You need only describe a portion of the larger circle.



8. Draw an arc of a circle of $4\frac{3}{8}''$ radius, and mark two points A, B on it $4''$ apart. Construct tangents at A, B *without using the centre of the circle.* [1894.]

XII.

1. Copy the given figure.*

2. A man starts from a point A, and walks 3 miles 1 furlong in a given direction. He then turns to his right at right angles, and walks $\frac{3}{4}$ of a mile. He then turns 60° to his left, and walks 3 miles 5 furlongs. He then

* Ex. XI., No. 32, was set.

turns back 130° to his right, and walks $4\frac{1}{2}$ miles. He then turns 70° to his right, and walks 2 miles 3 furlongs, arriving at *C*.

Plot his journey, and measure and write down the distance from *C* to *A*.

Scale—which must be constructed—$\frac{3}{4}$ of an inch to a mile.

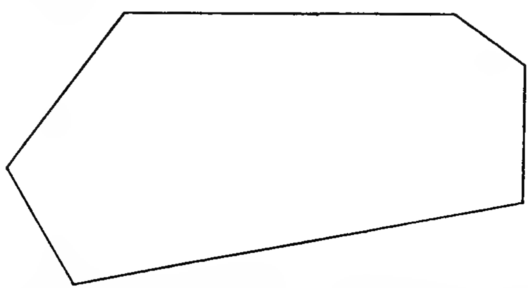
3. Construct a diagonal scale of Russian sagenes and feet. Make the scale to show 100 sagenes. Mark by two dots on the scale a length of 47 sagenes 3 feet. R.F. = $\frac{1}{1100}$.

Note.—The Russian foot is equal to the English foot, and 1 sagene = 7 feet.

4. From a given point *A* draw a parallel and also a perpendicular to a given straight line *BC*.

5. Construct a triangle of which the perimeter is 6 inches and the two base angles are 27° and 42°.

6. Construct a rectangle equal in area to the figure shown below, and having its length equal to three times the breadth.



7. A railway train is running in a straight line at 12 miles

an hour. The door of one of the carriages swings open, so that each point of the door describes with uniform velocity a quarter of a circle about the hinge, and immediately closes again, the whole time of opening and closing being half a second. The width of the door is 27 inches. Draw the curve traced out by a point on the edge of the door. Scale (which need not be constructed), $\frac{1}{2}$ inch = 1 foot. [1894.]

ANSWERS.

Chapter II. (Page 12.)

1. 2·37″.	2. 1·39″.	3. 2·07″.
4. 2·29″.	5. 2·31″.	6. 1·50″.
7. 2·05″.	8. 2·93″.	9. 3″.
10. 3·65″.	11. 1·10″; 2·20″	12. b = ·66″.
13. a = ·97″.	14. c = 1·57″.	15. 1·54″.
16. a = 1·19″.	17. a = 2·54″.	18. 1·50″
19. 1·67″.	20. 2·83″.	

Chapter III. (Page 25.)

1. 1·03″.	2. ·93″.	3. ·85″.
4. ·78″.	5. 1·46″.	6. 1·54″.
7. 1·70″.	8. 1·87″.	9. 1·15″.
10. ·96″.	11. ·83″.	12. 1·24″.
13. 2·49″.	14. ·69″.	15. ·83″.

Chapter IV. (Page 39.)

1. ·67″.	2. ·83″.	3. ·70″.
4. 1·46″.	5. (i.) ·54″.	5. (ii.) ·55″.
5. (iii.) ·34″.	6. ·67″.	7. 1·09″.
8. ·63″.	9. 1·15″.	10. ·67″.
11. ·55″.	12. 1·25″.	13. ·47″.
14. ·59″.	15. ·50″.	16. ·49″.
17. ·70″.	18. 2·51″.	19. 1·93″.

CHAPTER V. (Page 49.)

1. 5·48.
2. 1·5″.
6. 1·73″.
8. 3·09″ and 1·91″.
9. 3·24″.

CHAPTER VI. (Page 56.)

1. 5″. R.F. = $\frac{1}{126720}$.
2. 4·5″. $\frac{1}{80}$.
3. 5·28″.
4. 3·12″. $\frac{1}{202938}$.
5. 4·13″. $\frac{1}{459871}$.
6. 5·23″. $\frac{1}{6061}$.
7. 6·48″. $\frac{1}{44000}$.
8. 8·25″.
9. 5 miles = 6·88.″ $\frac{1}{46080}$.
10. 5 chains = 5″. $\frac{1}{792}$.
11. 20 poles = 5″.
12. 4 qrs. = 6″. No. R.F.
13. 2 months = 5·6″. No. R.F.
14. 1 hour = 5″. No. R.F.
15. 5·18″. $\frac{1}{6111}$.
16. 2,000 metres = 7·46″. $\frac{1}{10560}$.
17. 5 linear units = 7·5″.
18. 1,000 yards = 3·23″. $\frac{1}{11129}$.
19. 2,000 yards = 6·47″. $\frac{1}{11124}$.
20. $3\frac{1}{2}$ miles. 4 miles = 6″. $\frac{1}{42240}$.
21. 6·62 square feet.
22. $\frac{1}{23·6}$. 1·24 metres. 2·49 sq. m.
23. 1·91″.
24. 524 yards. 60°. 1,000 yards = 6·67″.
25. 128 feet. 200 feet = 6″.

CHAPTER VII. (Page 67.)

NOTE.—*Areas may be considered right if correct to the first place of decimals.*

1 and 2. 6·88 sq. ins.
3 and 4. 5·85 sq. ins.
5 and 6. 8·18 sq. ins.
9. 3·85 sq. ins.
10. 3·78 sq. ins.

Chapter VIII. (Page 73.)

1. 2·51″.	2. 2·01″.	3. 4·24.
4. 5·48.	5. 4·80.	6. 1·50.
7. 1·13.	8. 1·33.	9. 1·65″.
10. 1·36″.	11. 1·56″.	12. 1·24″.
13. 3·38.	14. 2·95.	15. 2·82.
16. 2·83.	17. 3·97.	18. 1·70.

Chapter IX. (Page 87.)

1. Prob. 1.	2. Prob. 2.
3. Prob. 2.	4. Prob. 6.
5. Prob. 6.	6. Prob. 7.
7. Prob. 7.	8. *See* Prob. 6 and p. 82.
9. *See* Prob. 9. and p. 82.	10. Prob. 9.
11. Prob. 10.	12. Prob. 10.
13. Prob. 3.	14. Prob. 11.
15. Prob 11.	16. Prob. 11.
17. Prob. 4.	18. Prob. 4.
19. Prob. 4.	20. Prob. 4.
21. Prob. 4.	

Chapter X. (Page 104.)

1. 2·55″.	2. 1·34″.	3. 77° or 103°.
4. 2·92 sq. ins.	5. 3·2″.	6. 1·67″.
7. 1·73″.	8. 1·64″.	9. ·84″.
10. ·84″.	11. 1·38 sq ins.	12. 1·34″.
13. 3·04″.	14. 2·35″.	15. 1·73″.
16. 2·01″.	17. 1·52″.	18. 1·87″.

ANSWERS TO PAPERS.

NOTE.—*Questions to which no reference is given are problems which will be found in the First Part.*

PAPER I.

1. 5·91″. $\frac{1}{536123}$.
4. 6·78″. $\frac{1}{2903}$.
5. Chap. IX., Prob. 7.
6. Chap. XII., Prob. 16.

PAPER II.

1. 5·70″.
2. Trisecting lines = 3·46″.
3. Chap. IX., Prob. 3.
4. 6·33″. $\frac{1}{10560}$.
5. Chap. X., Prob. 9. Side = 3·04″.

PAPER III.

2. 2,000 metres = 7·46″. $\frac{1}{10560}$.
3. 5 units = 7·5″.
6. Chap. X., Prob. 1. Longer side = 3·56″.

PAPER IV.

2. 7·02″. $\frac{1}{47892}$.
3. 6·32″.
4. AF is $3\frac{3}{8}$″. AB = ·44″.
6. 4·20″. Euc. VI. 1.

PAPER V.

2. 6·00″. $\frac{1}{3000}$.
3. 7·01″.
4. Chap. X., Prob. 5.
5. Chap. I., Prob. 4.

Paper VI.

2. 30 statute miles = 6·00″; 20 geog. miles = 4·61″. $\frac{1}{316800}$.
3. Chap. IX., Prob. 7. 4. Chap. II., Prob. 2.
5. 5·45″.
6. Chap. VII., Prob. 1. Area = 8·05 sq. ins.

Paper VII.

2. $\frac{1}{1223}$. 200 yards = 5·89″. 3. 50 Milan miles = 6·42″.
4. Chap. II., page 9. Side of triangle = 1·73″; side of first square = 1·14″; side of second square = 1·06″.
5. Use Chap. I., Prob. 4.

Paper VIII.

3. Divide *AB* in the proportion 5 : 12 : 7; radii = ·73″ and 1·02″.
4. Chap. II., page 9. Base = ·64″.
5. Chap. X., Prob. 10. Shorter side = ·88″.
6. 6 furlongs = 6·75″. 2,000 paces = 8·52″.
7. 1 li = 10·97″. $\frac{1}{2076}$.

Paper IX.

2. 10 cables = 6·09″. 3. 11 kilometres = 7·86″. $\frac{1}{5512}$.
5. Chap. I., Prob. 2. 6. Area = 10·55 sq. ins.
7. Chap. X., Prob. 1; 5·24″ or 1·04″.

Paper X.

2. 6·79″. 3. 128 feet.
6. If altitude of pentagon = altitude of equivalent triangle, then altitude of constructed triangle = 2·18″.
8. Chap. XII., Probs. 3, 11 and 5.

PAPER XI.

2. 2 yards = 6·55″.
3. 6·88″. $\frac{1}{46080}$.
4. Chap. III., Prob. 2.
5. Chap. II., page 9.
6. Reduce the two triangles to one of equal area. Then, with same altitude, base of required triangle = 3·16″. (Euc. VI. 1.)
7. The circles touch the tangent at *C*.
8. Chap. IX., Prob. 2.

PAPER XII.

2. 6 miles = 4·50″; *AC* = 2 m. 2 f.
3. 7·64″.
5. Chap. II., page 9; longest side = 2·72″.
6. Chap. X., Prob. 10. Rectangle = 1″ × 3″.
7. *See* Exercises XII., No. 33.

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